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Performance Evaluation of Extended EWMA Chart for AR Model with Exogenous Variables

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Abstract

The extended exponentially weighted moving average (Extended EWMA) control chart is an effective statistical process control method for monitoring and identifying shifts in process mean, particularly when dealing with autocorrelated data. One key performance measure used to evaluate the capability of control charts in detecting changes is the average run length (ARL). The primary goal of this study is to present the explicit formulas for calculating the ARL of the extended EWMA control chart for autoregressive models with exogenous variables (ARX) and exponential white noise. Another purpose is to compare the performance of the extended EWMA and the classical EWMA control charts under various conditions. The explicit formulas are derived from the ARL integral equation, which is expressed by the Fredholm integral equation. The accuracy of the exact solutions has been verified using the numerical integral equation (NIE) methods that employ four different composite quadrature rules. The result shows that the ARL values obtained from both methods are similar, and the computation time for the proposed explicit formulas is less than 0.001 second. In comparing the two control charts, it is evident that the extended EWMA control chart outperforms the traditional control chart in detecting shifts in the process mean, as confirmed by various overall performance criteria. Additionally, two real datasets, namely SCB stock price and GDP percentage expansions, are applied to demonstrate the effectiveness of the relevant control charts.

Keywords: Average Run Length; Extended EWMA Chart; Explicit Formula; Autoregressive with Exogenous Variables.

1. Introduction

Statistical Process Control (SPC) is a powerful set of problem-solving methods primarily used in the manufacturing industry to maintain and improve the quality of processes and products by reducing variability. A key visual tool in SPC is the control chart, which is extensively used to monitor process stability and detect special-cause variations or unnatural shifts in process parameters, such as mean and variance. These shifts can lead to the production process becoming out of control. The faster a control chart responds to changes, the quicker the process can be addressed and brought back into a controlled state. The concept of the traditional Shewhart chart, introduced by Shewhart [1], is classified as a memory-less control chart. One significant limitation of memory-less charts is their ineffectiveness in detecting minor changes. To overcome this, memory-type control charts, such as the cumulative sum (CUSUM) control chart [2] and the exponentially weighted moving average (EWMA) control chart [3], were developed to rapidly identify small to moderate variations in processes. Subsequently, several researchers have proposed enhanced control charts. For instance, Patel & Divecha [4] presented the modified exponentially weighted moving average (MEWMA) control chart to detect small shifts in process mean, and Khan et al. [5] improved upon this with a generalized form of MEWMA. Abbas et al. [6] combined the CUSUM and EWMA control charts, demonstrating that the mixed CUSUM-EWMA control chart performs better than either individual chart. In 2018, Naveed et al. [7] developed a new design for an EWMA-based

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statistic called the extended EWMA control chart, which showed greater sensitivity in monitoring small changes compared to the classical EWMA control chart. The extended EWMA scheme has since been studied to assess its performance in various simulated and practical situations [8, 9].

The fundamental assumption underlying traditional control charts is that the observations are independent and identically distributed. However, in real applications, successive samples from many processes are often dependent on time intervals and exhibit serial correlation. This dependence can negatively impact the performance of standard control charts, leading to incorrect indications and conclusions [10, 11]. To address the issue of autocorrelated data, researchers have investigated alternative strategies. One such approach involves fitting an appropriate time series model and then applying the uncorrelated property of the residuals, or white noise process, in the statistical control chart procedure [12, 13]. The time series autoregressive (AR) and moving average (MA) models, as well as the models comprising AR and MA, are usually employed for modeling and predicting autocorrelated data that depend on themselves. In some cases, the independent factors can impact the behavior of the process, improving prediction accuracy. Consequently, the time series models incorporated with the exogenous variables, such as ARX, MAX, or ARMAX, have been studied across various fields [14]. Regarding the random error of time series models known as white noise, which usually follows normal distribution, however, the white noise can also exhibit an exponential distribution [15].

Generally, the performance and sensitivity of control charts for monitoring and detecting variation in the process are typically assessed through the average run length, or ARL. This measure indicated the expected number of in-control observations before an out-of-control signal is detected. There are two types of ARL; ARL_0 refers to the average run length for an in-control process with no changes. Ideally, this value should be large, indicating that the control chart is stable and effective. ARL_1 denotes the average run length when the process is out of control and reflects the detection capabilities for various magnitudes of shifts. A smaller value ARL_1 is desirable, as it shows that the control chart can quickly identify any out-of-control conditions. Evaluating the ARL values is a crucial aspect when studying and developing control charts, as it allows for comparison of their capability. Previous research has employed different methodologies to calculate ARL values. For example, Champ & Rigdon [16] studied and compared the Markov Chain and the numerical integral equation (NIE) method for calculating the ARL of quality control charts. Naveed et al. [7] utilized Monte Carlo simulation for assessing the ARL and the proposed extended EWMA control chart. Nevertheless, the mentioned approaches can be time-consuming in terms of calculations.

Several researchers have investigated the derivation of the Average Run Length (ARL) integral equation under conditions of autocorrelation, particularly when the white noise process follows an exponential distribution, leading to the establishment of explicit formulas. Paichit [17] derived an exact solution for the ARL of the Cumulative Sum (CUSUM) control chart for an Autoregressive (AR) process with one exogenous variable (ARX(1)), where the white noise is characterized by an exponential distribution. The accuracy of the ARL was confirmed with Numerical Integration Evaluation (NIE) using the Gauss-Legendre rule, showing excellent agreement. Phanyaem [18] also presented an explicit formula for the ARL, comparing the accuracy of this formula against the NIE method using different quadrature rules for the CUSUM control chart when the observations belong to a seasonal ARX model with exponential white noise. The ARL from the explicit formula closely matched the NIE results, with an absolute percentage difference of less than 1%. Suriyaket & Phetcharat [19] developed an explicit formula for the ARL of the Maximum (MAX) process operating on an Exponentially Weighted Moving Average (EWMA) chart using techniques from Fredholm integral equations. They applied numerical integration methods, including Gaussian, midpoint, and trapezoidal rules, to verify the accuracy of the explicit formula. The results indicated that the ARL derived from their proposed method approximated the NIE results and outperformed the numerical methods in terms of computational time. Supharakonsakun [20] derived the ARL for a modified EWMA control chart applied to a Seasonal Moving Average (SMA) of order q (SMA(q)), where the white noise is exponentially distributed. The findings showed good agreement between the explicit formula and the numerical integral equation method.

Karoon et al. [21] explored explicit formulas for the ARL of an extended EWMA control chart designed for a trend AR(p) model, comparing its accuracy to that of the NIE method. Zhang et al. [22] formulated explicit expressions for the ARL and Average Delay Time (ADT) for the CUSUM control chart associated with a seasonal SMA(Q)s model. The performance comparison between the results obtained from the explicit formulas and the numerical integration approach indicated that the explicit formulas considerably reduced computational time. Recently, Peerajit [23] introduced an analytical solution for calculating the ARL of a long-memory ARFIMA(1, d, 1)(1, D, 1)s process with exponential white noise operating on a CUSUM control chart. The NIE method was employed to verify the accuracy of this proposed approach. The results from both methods were in close agreement, but the time required for computing the ARL using the proposed method was significantly shorter. Sunthornwat et al. [24] suggested explicit formulas for the ARL of the Homogeneously Weighted Moving Average (HWMA) control chart based on an AR process. Phanthuna et al. [25] examined the explicit formula for the ARL of a double-modified exponentially weighted moving average (DMEWMA) control chart applied to an AR process. They compared the ARLs computed using the explicit formula and numerical integral equation method to validate the former. Finally, Phanyaem [26] developed an exact formula for computing the average run length in an EWMA control chart, specifically for a $SARX(P,r)_L$ model. The results showed that the average run length calculated using the proposed method is close with from the numerical integral equation method.

According to the efficiency of the extended EWMA control chart and the approaches to calculate the ARL that are mentioned above, we are interested in deriving the ARL integral equation of the extended EWMA control chart when the observation is in the pattern of AR process with an exogenous variable that has not been proposed before. Therefore, the aim of this study is to derive the explicit formulas of the ARL on the extended EWMA chart for the ARX model when the white noise follows an exponential distribution and compare it to the numerical integral equation method in four different composite quadrature rules. Moreover, the comparison of the sensitivity of detecting changes of the extended EWMA and the EWMA control chart is conducted under various conditions. The two real datasets are studied to assess the proposed explicit formula for the control charts and presented in this article.

2. Preliminaries

The definitions of the time series model and the control charts, including their characteristics, are given in this section.

2.1. Time Series ARX Model

Let Y_t be a sequence observation from the ARX(p,r) model defined as:

$$Y_t = \mu + \sum_{i=1}^{p} \phi_i Y_{t-i} + \varepsilon_t + \sum_{j=1}^{r} \beta_j X_{jt}\,, t = 1,2,3,\ldots \tag{1}$$

where $\boldsymbol{\mu}$ is a constant, $\boldsymbol{\phi_i} \in (-\mathbf{1},\mathbf{1})$ is an autoregressive coefficient, $\boldsymbol{X_{jt}}$ is an exogenous variable, $\boldsymbol{\beta_j}$ is a coefficient of $\boldsymbol{X_{jt}}$ and $\boldsymbol{\varepsilon_t}$ is an error term or a white noise process assumed to follow the exponentially distributed, therefore, $\boldsymbol{\varepsilon_t \sim Exp(\alpha)}$.

2.2. Extended EWMA Control Chart

The extended EWMA statistic improved by Naveed et al. [7] can be defined by the recursive equation:

$$E_t = \lambda_1 Y_t - \lambda_2 Y_{t-1} + (1 - \lambda_1 + \lambda_2)E_{t-1}, \qquad t = 1, 2, 3, \ldots. \tag{2}$$

where Y_t is a sequence observation from the ARX process, $\lambda_1 \in (0,1]$ and $\lambda_2 \in [0, \lambda_1)$ are smoothing constants. The upper control limit (UCL) and the lower control limit (LCL) are:

$$\begin{aligned} \text{UCL} &= \mu_0 + \omega\sigma\sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1 - \lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2)^2}} \\ \text{LCL} &= \mu_0 - \omega\sigma\sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1 - \lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2)^2}} \end{aligned} \tag{3}$$

where μ_0 is a target mean, σ is a process standard deviation and ω is an appropriate control limit width.

The stopping time for the extended EWMA control chart is $\tau_{a,b} = inf\{t > 0: E_t < a \cup E_t > b\}$ where a and b represent the LCL and UCL, respectively.

The extended EWMA control chart converts to the classical EWMA scheme, $Z_t = \lambda_1 Y_t + (1 - \lambda_1)Z_{t-1}$ when $\lambda_2 = 0$. Similarly, the LCL (a') and UCL (b') of the EWMA control chart can be determined by equation (3) when$\lambda_2 = 0$ with constant width $\omega = \omega_c$, therefore, the stopping time for the EWMA control chart is $\tau_{a',b'} = inf\{t > 0: Z_t < a' \cup Z_t > b'\}$.

2.3. Average Run Length

Let $\varepsilon_t, t = 1,2,\ldots$ be a sequence of independent random variables with a probability density function $f(w, \alpha)$ where $\alpha$is the parameter. The in-control state is normally with the parameter $\alpha = \alpha_0$ and assumed that there is no change in the process. On the contrary, the parameter $\alpha = \alpha_1$ when the process has changed to out-of-control state at the change-point time, φ. Average run length or ARL is the common characteristic of control charts to measure and compare their performance in detecting changes in parameters. Ideally, the ARL for in-control process denoted as ARL_0 are required to be sufficiently large in order to reduce the number of false out-of-control signals, whereas, the ARL for out-of-control state or ARL_1 must be small to quickly detect a correct out-of-control signal. In this study, the stopping time $(\tau_{a,b})$ are used as the alarm signals, therefore, the ARL is defined as:

$$ARL = \begin{cases} ARL_0 = \hat{E}_\varphi(\tau_{a,b}), \varphi = \infty \\ ARL_1 = \hat{E}_\varphi(\tau_{a,b} | \tau_{a,b} \geq 1), \varphi = 1 \end{cases} \tag{4}$$

where $\hat{E}_{\varphi}$ is the expectation of the stopping time under the assumption that the change-point time occur at time, φ. ARL_0 represents the in-control ARL which implies that the change-point time does not exist, whereas ARL_1 denotes the out-of-control ARL when the change point appears at the first time.

3. ARL Evaluation Methods

In this section, the explicit formulas of ARL for the extended EWMA control chart of the ARX(p,r) model with exponential white noise derived from a Fredholm integral equation of the second kind are presented. Moreover, an approximated ARL from the numerical integral equation (NIE) method is used to confirm the accuracy of the exact solutions.

Here, the in-control state of extended EWMA scheme for ARX(p,r) model can be rewritten in the form of white noise process ε_t as:

$$\left[\frac{a-(\lambda_1\phi_1-\lambda_2)Y_0-(1-\lambda_1+\lambda_2)\nu}{\lambda_1}-\mu-\sum_{i=2}^{p}\phi_i Y_{t-i}-\sum_{j=1}^{r}\beta_j X_{jt}\right] < \varepsilon_t < \left[\frac{b-(\lambda_1\phi_1-\lambda_2)Y_0-(1-\lambda_1+\lambda_2)\nu}{\lambda_1}-\mu-\sum_{i=2}^{p}\phi_i Y_{t-i}-\sum_{j=1}^{r}\beta_j X_{jt}\right] \tag{5}$$

where ν is an initial value of the extended EWMA control chart and Y_0 is an initial value of ARX(p,r) model.

Let $\partial(\nu)$ be an ARL of an initial value ν which can be described by applying the Fredholm integral equation second kind according to the method of Champ & Rigdon [16] of as follows:

$$\partial(\nu) = 1 + \int_{\frac{a-(\lambda_1\phi_1-\lambda_2)Y_0-(1-\lambda_1+\lambda_2)\nu}{\lambda_1}-\mu-\sum_{i=2}^{p}\phi_i Y_{t-i}-\sum_{j=1}^{r}\beta_j X_{jt}}^{\frac{b-(\lambda_1\phi_1-\lambda_2)Y_0-(1-\lambda_1+\lambda_2)\nu}{\lambda_1}-\mu-\sum_{i=2}^{p}\phi_i Y_{t-i}-\sum_{j=1}^{r}\beta_j X_{jt}} \partial(E_t)\, f(\varepsilon_t)\, d\varepsilon_t \tag{6}$$

After setting new variables, then, the $\partial(\nu)$ can be defined as

$$\partial(\nu) = 1 + \frac{1}{\lambda_1}\int_a^b \partial(w) f\left(\frac{w-(\lambda_1\phi_1-\lambda_2)Y_0-(1-\lambda_1+\lambda_2)\nu}{\lambda_1} - \mu - \sum_{i=2}^{p}\phi_i Y_{t-i} - \sum_{j=1}^{r}\beta_j X_{jt}\right) dw \tag{7}$$

White noise process is assumed to be random variables exponentially distributed so that the pdf is $f(w) = \frac{1}{\alpha}e^{\frac{-w}{\alpha}}$, therefore, the ARL can be rearranged as the following equation:

$$\partial(\nu) = 1 + \frac{1}{\lambda_1}\int_a^b \partial(w) e^{\frac{-w+(\lambda_1\phi_1-\lambda_2)Y_0+(1-\lambda_1+\lambda_2)\nu}{\alpha\lambda_1}+\frac{\mu+\sum_{i=2}^{p}\phi_i Y_{t-i}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}} dw \tag{8}$$

The exact and the approximated solutions of the integral Equation 8 are revealed in the next subsections.

3.1. The Proposed Explicit Formula

From the integral Equation 8, we can rewrite in Equation 9:

$$\partial(\nu) = 1 + \frac{C(\nu)}{\alpha\lambda_1}D \tag{9}$$

where $C(\nu) = e^{\frac{(\lambda_1\phi_1-\lambda_2)Y_0+(1-\lambda_1+\lambda_2)\nu}{\alpha\lambda_1}+\frac{\mu+\sum_{i=2}^{p}\phi_i Y_{t-i}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}$ and $D = \int_a^b \partial(w) e^{\frac{-w}{\alpha\lambda_1}} dw$. Then, consider the variable D in this form;

$$D = \int_a^b \left(1 + \frac{C(w)}{\alpha\lambda_1}D\right) e^{\frac{-w}{\alpha\lambda_1}} dw = \int_a^b e^{\frac{-w}{\alpha\lambda_1}} dw + \frac{D}{\alpha\lambda_1}\int_a^b e^{\frac{(\lambda_1\phi_1-\lambda_2)Y_0+(1-\lambda_1+\lambda_2)w}{\alpha\lambda_1}+\frac{\mu+\sum_{i=2}^{p}\phi_i Y_{t-i}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}-\frac{w}{\alpha\lambda_1}} dw = \frac{-\alpha\lambda_1\left(e^{\frac{-b}{\alpha\lambda_1}}-e^{\frac{-a}{\alpha\lambda_1}}\right)}{1+\frac{1}{\lambda_1-\lambda_2}e^{\frac{(\lambda_1\phi_1-\lambda_2)Y_0}{\alpha\lambda_1}+\frac{\mu+\sum_{i=2}^{p}\phi_i Y_{t-i}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}\left(e^{\frac{-(\lambda_1-\lambda_2)b}{\alpha\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)a}{\alpha\lambda_1}}\right)} \tag{10}$$

After substituting D into Equation 9, we finally have the explicit formula for in-control state as,

$$\partial(\nu) = 1 - \frac{(\lambda_1-\lambda_2)\left(e^{\frac{-b}{\alpha_0\lambda_1}}-e^{\frac{-a}{\alpha_0\lambda_1}}\right)e^{\frac{(1-\lambda_1+\lambda_2)\nu}{\alpha_0\lambda_1}}}{(\lambda_1-\lambda_2)e^{\frac{-(\lambda_1\phi_1-\lambda_2)Y_0}{\alpha_0\lambda_1}}e^{\frac{-\left(\mu+\sum_{i=2}^{p}\phi_i Y_{t-i}+\sum_{j=1}^{r}\beta_j X_{jt}\right)}{\alpha_0}}+\left(e^{\frac{-(\lambda_1-\lambda_2)b}{\alpha_0\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)a}{\alpha_0\lambda_1}}\right)} \tag{11}$$

and the out-of-control exact solution for ARL as;

$$\partial(v) = 1 - \frac{(\lambda_1-\lambda_2)\left(e^{\frac{-b}{\alpha_1\lambda_1}}-e^{\frac{-a}{\alpha\lambda_1}}\right)e^{\frac{(1-\lambda_1+\lambda_2)v}{\alpha_1\lambda_1}}}{(\lambda_1-\lambda_2)e^{\frac{-(\lambda_1\phi_1-\lambda_2)Y_0}{\alpha_1\lambda_1}}e^{\frac{-\left(\mu+\sum_{i=2}^{p}\phi_i Y_{t-i}+\sum_{j=1}^{r}\beta_j X_{jt}\right)}{\alpha_1}}+\left(e^{\frac{-(\lambda_1-\lambda_2)b}{\alpha_1\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)a}{\alpha_1\lambda_1}}\right)} \quad (12)$$

Furthermore, Banach's fixed point theorem from mathematical analysis is utilized to confirm the existence and uniqueness of the proposed explicit formula which is the solution to the ARL integral equation.

Definition 1. Let (S,Δ)be a metric space. An operator $M: S \to S$is a contractive mapping or a contraction if there exists a constant $\kappa \in (0,1)$such that $\Delta\big(M(s_1), M(s_2)\big) \le \kappa\Delta(s_1,s_2)$for all s_1,$s_2$in S.

Theorem 1. Banach's fixed point theorem:

Let (S,Δ)be a complete metric space and $M: S \to S$be a contraction on S. Then M has a unique fixed point such that $M(s) = s, s \in S$.

In this present work, we consider the ARL Equation 8 in the set of all continuous function denoted by $C[a,b]$. Consequently, the space $(C[a,b], \|.\|_\infty)$ is complete with a norm given by $\|\partial(v)\|_\infty = \sup_{v\in[a,b]}|\partial(v)|$. The following theorem and its proof provide the second condition of Theorem 1 which can imply that the ARL integral equation has an only one solution.

Theorem 2. Let $M: C[a,b] \to C[a,b]$ be an operator defined as,

$$M\big(\partial(v)\big) = \partial(v) = 1 + \frac{1}{\alpha\lambda_1}\int_a^b \partial(w)k(v,w)dw \quad (14)$$

where $k(v,w) = e^{\frac{-w+(\lambda_1\phi_1-\lambda_2)Y_0+(1-\lambda_1+\lambda_2)v}{\alpha\lambda_1}+\frac{\mu+\sum_{i=2}^{p}\phi_i Y_{t-i}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}$ is a kernel function. Then, M is a contraction.

Proof. Let $\partial_1(v)$ and $\partial_2(v)$ are two arbitrary functions in $C[a,b]$, then, consider;

$$\|M(\partial_1(v)) - M(\partial_2(v))\|_\infty = \sup_{v\in[a,b]}\left|\int_a^b |\partial_1(v)-\partial_2(v)|dw\right| \le \sup_{v\in[a,b]}\int_a^b |k(v,w)|\,|\partial_1(v)-\partial_2(v)|dy$$
$$\le \sup_{v\in[a,b]}\int_a^b |k(v,w)|\,dw\|M(\partial_1(v)) - M(\partial_2(v))\|_\infty = \kappa\|M(\partial_1(v)) - M(\partial_2(v))\|_\infty$$

where $\kappa < 1$and $\kappa = \sup_{v\in[a,b]}\int_a^b |k(v,w)|\,dw$is a positive constant. This implies that $M$is a contraction.

3.2. Numerical Integral Equation Method

The NIE method for approximating a solution of an integral equation is the use of a quadrature rule which determined by the set of nodes or points, $\{x_j, j = 0,1,\dots,m\}$ obtained from the partition of an integral limit $[a,b]$ into$m$subintervals and the set of weights, $\{w_j, j = 0,1,\dots,m\}$, generally, the approximation of an integral can be expressed as $\int_a^b W(x)f(x)dx \approx \sum_{j=1}^{m} w_j f(x_j)$.

The integral equation to evaluate the ARL in (8) can be estimated by the solution of m linear equation systems,

$$\tilde{\partial}(x_i) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j\partial(x_j)\, f\left(\frac{x_j-(\lambda_1\phi_1-\lambda_2)Y_{t-1}-(1-\lambda_1+\lambda_2)x_i}{\lambda_1} - \mu - \sum_{i=2}^{p}\phi_i Y_{t-i} - \sum_{j=1}^{r}\beta_j X_{jt}\right), i = 1,\dots,m \quad (15)$$

The system of the m linear equations is $L_{m\times 1} = (I_m - R_{m\times m})^{-1}1_{m\times 1}$ where $L_{m\times 1} = [\tilde{L}(a_1) \quad \tilde{L}(a_2) \quad \dots \quad \tilde{L}(a_m)]^T$.

Let $R_{m\times m}$ be a matrix and define the m to m^{th} as elements of matrix R as follows,

$$[R_{ij}] \approx \frac{1}{\lambda_1} w_j f\left(\frac{x_j-(\lambda_1\phi_1-\lambda_2)Y_{t-1}-(1-\lambda_1+\lambda_2)x_i}{\lambda_1} - \mu - \sum_{i=2}^{p}\phi_i Y_{t-i} - \sum_{j=1}^{r}\beta_j X_{jt}\right) \quad (16)$$

Finally, the general numerical approximation of $\partial(v)$ is expressed as:

$$\tilde{\partial}(v) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j\partial(x_j)\, f\left(\frac{x_j-(\lambda_1\phi_1-\lambda_2)Y_{t-1}-(1-\lambda_1+\lambda_2)v}{\lambda_1} - \mu - \sum_{i=2}^{p}\phi_i Y_{t-i} - \sum_{j=1}^{r}\beta_j X_{jt}\right) \quad (17)$$

The details of different composite quadrature rules including the location of nodes and their weights when setting the equal width $h = (b-a)/m$ and $K_j = \frac{x_j-(\lambda_1\phi_1-\lambda_2)Y_{t-1}-(1-\lambda_1+\lambda_2)v}{\lambda_1} - \mu - \sum_{i=2}^{p}\phi_i Y_{t-i} - \sum_{j=1}^{r}\beta_j X_{jt}$are presented in Table 1.

Table 1. The composite quadrature rules

Composite rules	Equation	Node(x_j)	Weight (w_j)
Midpoint	$\tilde{\partial}_M(v) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j L(x_j) f(K_j)$	$a + \left(j - \frac{1}{2}\right)h$	h
Trapezoidal	$\tilde{\partial}_T(v) = 1 + \frac{1}{\lambda_1}\sum_{j=0}^{m} w_j L(w_j) f(K_j)$	$a + jh$	$\frac{h}{2}; j = 0, m, h; j = 1, \ldots, m-1$
Simpson's	$\tilde{\partial}_S(v) = 1 + \frac{1}{\lambda_1}\sum_{j=0}^{2n} w_j L(x_j) f(K_j)$ where m=2n	$a + jh$	$\frac{h}{3}; j = 0{,}2n, \frac{4h}{3}; j = 1, .., 2n-1, \frac{2h}{3}; j = 2, .., 2n-2$
Bool's	$\tilde{\partial}_B(v) = 1 + \frac{1}{\lambda_1}\sum_{j=0}^{4n} w_j L(x_j) f(K_j)$ where m=4n	$a + jh$	$\frac{14h}{45}; j = 0{,}4n, \frac{64h}{45}; j = 1, \ldots, 4n-3{,}4n-1$ $\frac{24h}{45}; j = 2, \ldots, 4n-2, \frac{28h}{45}; j = 4, \ldots, 4n-4$

4. Simulation Result

The details of simulation study, performance criteria and results for verifying the accuracy of the proposed explicit formula to assess the ARL of ARX(p,r) process running on the extended EWMA control chart are provided in subsection 4.1. The explicit formula and NIE method to evaluate the ARL were computed by the Mathematica program in the 64-bit operating system, AMD Ryzen 7 4700U with Radeon Graphics 2.00 GHz processor. In addition, the performance comparisons of the extended EWMA control chart and the classical EWMA control chart in detecting process mean change under different conditions are presented in subsection 4.2. The real datasets in finance and economics fields are studied and revealed in 4.3.

4.1. The Accuracy of the Proposed Explicit Formula

The numerical algorithm for calculating the ARL can be concluded as the following steps.

Step 1: Set the values of

- The autoregressive coefficients (ϕ_i), the coefficient exogenous variables (β_j), constant (μ), the initial value of autoregressive: $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p}$ and the exogenous variables (X_{jt}) in the ARX(p,r) model.
- The smoothing constants $(\lambda_1, \lambda_2)$and the initial value of the extended EWMA control chart $(E_0 = v)$.
- The exponential white noise parameter for in-control state, α_0.
- The shifts value, δ = 0.005, 0.01, 0.025, 0.05, 0.1, 0.25, 0.5, 1 to determine the out-of-control state parameter $\alpha_1 = (1+\delta)\alpha_0$.
- An acceptable ARL_0 = 370 for in-control state and the lower control limit a.

Step 2: Compute the upper control limit, b by Equation 11 that yield the desire average run length for in - control process.

Step 3: Compute a solution of ARL_1 for the specific shift in process where $\alpha_1 = (1+\delta)\alpha_0$ by the equation of explicit formula (12) and the NIE method (17) with different quadrature rules and set the numbers of subinterval, $m = 600$. The CPU time of each method is also collected.

Step 4: Compute the absolute percentage difference, APD (%) which is defined as:

$$APD(\%) = \frac{|\partial(v) - \tilde{\partial}(v)|}{\partial(v)} \times 100 \quad (18)$$

Table 2 presents the ARL values of the extended EWMA control chart, calculated using an explicit formula and four composite quadratic rules for the NIE method. This analysis was conducted on various ARX(p,r) processes, specifically the ARX(1,2), ARX(2,1), and ARX(3,2) models when $a = 0$, $\boldsymbol{\mu} = \mathbf{1}$ and specific $\boldsymbol{\lambda_1} = \mathbf{0.05}$ and $\lambda_2 = 0.025$.The results indicate that the ARL values obtained from the derived explicit formula are very similar to those approximated by the NIE method. In fact, the small APD (%) suggests that the proposed explicit formula can accurately evaluate the ARL when compared to the NIE method. In addition, the CPU time shows that the explicit formula takes less than 0.001 seconds to compute the ARL, whereas the NIE method takes approximately 3.1 to 3.5 seconds. Notably, the composite Bool's rule is the fastest among the other rules. The advantage of explicit formula in this work which rapidly calculating the accurate ARL values is similar to the previous studies [20-25] that derived the explicit formula for other control charts with various pattern of time series model with exponential white noise.

4.2. The Performance of Extended EWMA Control Chart

The ARL of the extended EWMA control chart has been studied under different conditions of the relevant parameters to assess the sensitivity in detecting the process change. The overall performance measures namely the average extra quadratic loss (AEQL), the performance comparison index (PCI) and the relative mean index (RMI) are used to compare the efficiency of the extended EWMA control chart and the classical EWMA control chart and defined as follows:

$$AEQL = \frac{1}{\Delta}\sum_{\delta_i=\delta_{min}}^{\delta_{max}\Sigma}\left(\delta_i^2 \times ARL(\delta_i)\right) \tag{19}$$

where δ_i is the value of change in the process mean at each level i, $ARL(\delta_i)$ is the ARL value of the control chart for the change level $\delta_i$and Δ is the number of shift levels from δ_{min} to δ_{max}. In this study, the increments Δ =9 from δ_{min}=0 to δ_{max}=1. The control chart with the smallest value of AEQL is implied to be the most effective one.

The PCI is the ratio of the AEQL of a control chart and the AEQL of the most effective control chart denoted as $\text{AEQL}_{\text{base}}$ defined as,

$$PCI = \frac{AEQL}{AEQL_{base}} \tag{20}$$

The RMI is calculated as:

$$RMI = \frac{1}{n}\sum_{i=1}^{n}\frac{ARL(\delta_i)-ARL_{smallest}(\delta_i)}{ARL_{smallest}(\delta_i)} \tag{21}$$

where n is the number of the shifts, $ARL(\delta_i)$, $i = 1,\dots,n$is the ARL of a control chart for a shift δ_i and $ARL_{smallest}(\delta_i)$ is the smallest ARL among the competing control charts for the shift δ_i. It is similar to the AEQL which can implies that a control chart with a lowest value of RMI has the most powerful detection ability.

Table 3 reveals the ARL values of ARX(1,1) model when λ_1= 0.05, 0.10, 0.15 and λ_2= 0.015, 0.025, 0.035, 0.045. The result shows that the extended EWMA charts with different values of λ_2 obtained the lower ARL than the EWMA charts for all magnitudes of change. However, it can be seen that the ARL values of the extended EWMA and classical EWMA chart are hardly different when the shift sizes are larger. The AEQL, PCI and RMI values indicate that the performance of the extended EWMA control chart slightly improved when λ_2 increased. Moreover, we consider the AEQL values of each fixed $\lambda_2$and found that the control chart showed the better performance when λ_1 is increased. Similary, Table 4 illustrate the ARL values of ARX(2,2) model when the smoothing parameterλ_1= 0.05, 0.10, 0.15 and λ_2=0.3λ_1, 0.5λ_1, 0.7λ_1, 0.9λ_1. The results confirm that the extended EWMA control charts quicklier detecting changes than the EWMA control chart and also show the better performance when $\lambda_2$increased and close to λ_1.

Table 2. The ARL from explicit formula against NIE method using four quadrature rules for the extended EWMA control chart on ARX(*p*,*r*) model given $a = 0$, $\mu = 1$, $\lambda_1 = 0.05$, $\lambda_2 = 0.025$ and ARL₀=370

ARX ϕ_i, β_j, b	δ	Explicit (CPU Time)	NIE (CPU Time in seconds, APD(%)) Midpoint	Trapezoidal	Simpson's	Bool's
ARX(1,2) ϕ_1= -0.2 β_1= 0.25 β_2= 0.10 b = 0.00029919	0.000	370.79588139338 (<0.001)	370.7958813921 (3.437, 3.500×10^{-10})	370.7958813968 (3.422, 9.304×10^{-10})	370.7958813937 (3.375, 7.687×10^{-11})	370.7958813937 (3.360, 7.714×10^{-11})
	0.005	138.81636527871 (<0.001)	138.8163652788 (3.453, 9.797×10^{-11})	138.8163652806 (3.484, 1.335×10^{-9})	138.8163652794 (3.438, 4.963×10^{-10})	138.8163652794 (3.437, 5.108×10^{-10})
	0.010	84.385613287935 (<0.001)	84.38561328855 (3.453, 6.699×10^{-10})	84.38561328957 (3.484, 1.940×10^{-9})	84.38561328889 (3.453, 1.131×10^{-9})	84.38561328889 (3.438, 1.132×10^{-9})
	0.025	37.566999643581 (<0.001)	37.56699964348 (3.563, 2.705×10^{-10})	37.56699964391 (3.484, 8.853×10^{-10})	37.56699964362 (3.437, 1.147×10^{-10})	37.56699964362 (3.438, 1.147×10^{-10})
	0.050	18.548715459259 (<0.001)	18.54871545920 (3.547, 2.971×10^{-10})	18.54871545940 (3.453, 7.720×10^{-10})	18.54871545927 (3.438, 5.930×10^{-11})	18.54871545927 (3.437, 5.930×10^{-11})
	0.100	8.5084421809348 (<0.001)	8.508442180907 (3.437, 3.267×10^{-10})	8.508442180984 (3.469, 5.807×10^{-10})	8.508442180933 (3.453, 2.434×10^{-11})	8.508442180933 (3.438, 2.421×10^{-11})
	0.250	2.8749912056601 (<0.001)	2.874991205656 (3.532, 1.537×10^{-10})	2.874991205670 (3.485, 3.656×10^{-10})	2.874991205660 (3.453, 1.948×10^{-11})	2.874991205660 (3.437, 1.948×10^{-11})
	0.500	1.4971503983684 (<0.001)	1.497150398367 (3.500, 2.177×10^{-9})	1.497150398370 (3.468, 1.994×10^{-9})	1.497150398368 (3.438, 2.116×10^{-9})	1.497150398368 (3.437, 2.116×10^{-9})
	1.000	1.1054759084698 (<0.001)	1.105475908470 (3.484, 1.087×10^{-11})	1.105475908470 (3.484, 1.900×10^{-11})	1.105475908470 (3.438, 9.039×10^{-13})	1.105475908470 (3.438, 9.039×10^{-13})

ARX(2,1) ϕ_1= 0.1 ϕ_2= -0.2 β_1= 0.5 b = 0.004929596	0.000	370.00748430131 (<0.001)	370.0074838708 $(3.453, 1.163\times10^{-7})$	370.0074851591 $(3.422, 2.318\times10^{-7})$	370.0074843002 $(3.375, 2.876\times10^{-10})$	370.0074843002 $(3.391, 2.873\times10^{-10})$
	0.005	190.55222061678 (<0.001)	190.5522204017 $(3.468, 1.129\times10^{-7})$	190.5522210465 $(3.469, 2.254\times10^{-7})$	190.5522206167 $(3.438, 7.872\times10^{-11})$	190.5522206167 $(3.438, 7.872\times10^{-11})$
	0.010	127.43166667364 (<0.001)	127.4316665325 $(3.469, 1.107\times10^{-7})$	127.4316669558 $(3.484, 2.215\times10^{-7})$	127.4316666736 $(3.453, 1.724\times10^{-11})$	127.4316666736 $(3.422, 1.648\times10^{-11})$
	0.025	62.634879012605 (<0.001)	62.63487894622 $(3.531, 1.060\times10^{-7})$	62.63487914542 $(3.453, 2.120\times10^{-7})$	62.63487901262 $(3.437, 2.699\times10^{-11})$	62.63487901262 $(3.422, 2.699\times10^{-11})$
	0.050	32.758067253609 (<0.001)	32.75806722111 $(3.438, 9.919\times10^{-8})$	32.75806731864 $(3.500, 1.986\times10^{-7})$	32.75806725362 $(3.453, 5.709\times10^{-11})$	32.75806725362 $(3.453, 5.709\times10^{-11})$
	0.100	15.823474648979 (<0.001)	15.82347463517 $(3.438, 8.739\times10^{-8})$	15.82347467659 $(3.500, 1.744\times10^{-7})$	15.82347464898 $(3.438, 1.416\times10^{-10})$	15.82347464898 $(3.422, 1.416\times10^{-10})$
	0.250	5.5103356099647 (<0.001)	5.510335606715 $(3.469, 5.897\times10^{-8})$	5.510335616464 $(3.500, 1.179\times10^{-7})$	5.510335609965 $(3.453, 2.354\times10^{-12})$	5.510335609966 $(3.438, 2.354\times10^{-12})$
	0.500	2.5546720917965 (<0.001)	2.554672091019 $(3.515, 3.044\times10^{-8})$	2.554672093352 $(3.468, 6.087\times10^{-8})$	2.554672091796 $(3.437, 3.824\times10^{-13})$	2.554672091795 $(3.438, 3.824\times10^{-13})$
	1.000	1.4769505548513 (<0.001)	1.476950554717 $(3.453, 9.083\times10^{-9})$	1.47695055512 $(3.484, 1.817\times10^{-8})$	1.476950554851 $(3.453, 6.765\times10^{-13})$	1.476950554851 $(3.453, 6.765\times10^{-13})$
ARX(3,2) ϕ_1= -0.1 ϕ_2= 0.2 ϕ_3= -0.3 β_1= 0.5 β_2=-0.25 b = 0.000347603	0.000	370.02826148995 (<0.001)	370.0282614895 $(3.516, 1.311\times10^{-10})$	370.0282614959 $(3.437, 1.597\times10^{-9})$	370.028261491595 $(3.422, 4.446\times10^{-10})$	370.028261491598 $(3.422, 4.454\times10^{-10})$
	0.005	140.84047889377 (<0.001)	140.840478892 $(3.484, 1.267\times10^{-9})$	140.8404788943 $(3.532, 4.040\times10^{-10})$	140.840478892769 $(3.422, 7.100\times10^{-10})$	140.84047889277 $(3.500, 7.093\times10^{-10})$
	0.010	85.969167561932 (<0.001)	85.96916756157 $(3.516, 4.169\times10^{-10})$	85.96916756298 $(3.516, 1.222\times10^{-9})$	85.9691675620432 $(3.484, 1.292\times10^{-10})$	85.9691675620434 $(3.469, 1.295\times10^{-10})$
	0.025	38.427560279249 (<0.001)	38.42756027909 $(3.547, 4.205\times10^{-10})$	38.42756027969 $(3.547, 1.141\times10^{-9})$	38.427560279287 $(3.516, 9.966\times10^{-11})$	38.4275602792872 $(3.485, 9.992\times10^{-11})$
	0.050	19.019467342452 (<0.001)	19.01946734238 $(3.515, 3.801\times10^{-10})$	19.01946734265 $(3.532, 1.065\times10^{-9})$	19.01946734247 $(3.484, 1.105\times10^{-10})$	19.01946734247 $(3.469, 1.105\times10^{-10})$
	0.100	8.7437261931818 (<0.001)	8.743726193149 $(3.640, 3.704\times10^{-10})$	8.743726193257 $(3.531, 8.588\times10^{-10})$	8.7437261931852 $(3.469, 3.923\times10^{-11})$	8.74372619318519 $(3.469, 3.935\times10^{-11})$
	0.250	2.9548002268956 (<0.001)	2.954800226896 $(3.485, 2.427\times10^{-10})$	2.95480022691 $(3.500, 4.680\times10^{-10})$	2.954800226896 $(3.484, 5.756\times10^{-12})$	2.954800226896 $(3.469, 5.756\times10^{-12})$
	0.500	1.5261219828111 (<0.001)	1.526121982810 $(3.500, 8.845\times10^{-11})$	1.526121982814 $(3.515, 1.684\times10^{-10})$	1.526121982811 $(3.484, 3.274\times10^{-12})$	1.526121982811 $(3.453, 3.274\times10^{-12})$
	1.000	1.1140149698577 (<0.001)	1.1140149698575 $(3.578, 1.437\times10^{-11})$	1.114014969858 $(3.562, 2.872\times10^{-11})$	1.1140149698578 $(3.484, 2.872\times10^{-11})$	1.114014969858 $(3.484, 2.872\times10^{-11})$

Table 3. The ARL for the extended EWMA control chart on ARX(1,1) model compare with the EWMA chart when $a = 0$, $\phi_1 = 0.3$ and $\beta_1 = 0.5$ are given

λ_1	Shift	λ_2				EWMA (b'= 0.02128157)
		0.015 (b = 0.00856734)	0.025 (b =0.004688704)	0.035 (b =0.002569308)	0.045 (b =0.001408792)	
0.05	0.000	370.0230749	370.020397	370.0022024	370.0094036	370.0106203
	0.005	202.5551198	189.3755089	177.7365048	167.4941544	225.9518908
	0.010	138.6254961	126.3611946	116.0110431	107.2478077	161.9505653
	0.025	69.96172424	61.9557122	55.51111554	50.26046147	86.40129664
	0.050	37.18100648	32.35704228	28.56364099	25.52882316	47.48474047
	0.100	18.23027117	15.61062999	13.58536505	11.98626866	23.98344592
	0.250	6.461339461	5.429878966	4.652293688	4.051757178	8.80793257
	0.500	2.979979859	2.520221731	2.185067614	1.93488656	4.069266324
	1.000	1.650467505	1.463568934	1.334456146	1.243486237	2.121210512
	AEQL	0.348578213	0.302898171	0.26994368	0.245605819	0.458157095
	PCI	1.419258775	1.233269522	1.09909318	1	1.865416289
	RMI	0.370374045	0.214558988	0.094476877	0	0.713546025

0.10		(b =0.0274078)	(b =0.02021706)	(b =0.014933349)	(b =0.01104089)	(b^*=0.04343651)
	0.000	370.08679	370.0148561	370.0069575	370.0150284	370.0130927
	0.005	126.6270088	119.6525792	113.4260721	107.8149375	138.9463404
	0.010	76.35379692	71.31980306	66.91788227	63.02495293	85.53792815
	0.025	34.8301204	32.20631679	29.95111527	27.98707599	39.74672435
	0.050	18.27777373	16.8175237	15.57170341	14.49398222	21.04399329
	0.100	9.419977003	8.63595675	7.970708904	7.39822652	10.91527025
	0.250	4.005211661	3.668840714	3.386090563	3.145134846	4.652867167
	0.500	2.285739767	2.112181538	1.968257802	1.847385373	2.624587041
	1.000	1.524507577	1.437949538	1.367737098	1.310163906	1.69751911
	AEQL	0.279859148	0.261550343	0.246479494	0.23392091	0.315900212
	PCI	1.196383633	1.118114422	1.053687307	1	1.350457349
	RMI	0.204350975	0.125239921	0.057967885	0	0.354353665
0.15		(b =0.04833234)	(b =0.03935632)	(b =0.03208744)	(b =0.02618639)	(b^*=0.06598734)
	0.000	370.0397208	370.0157474	370.0168905	370.0286903	370.0338381
	0.005	105.0220357	100.3838646	96.18482371	92.34918578	113.0463707
	0.010	61.35845538	58.22081974	55.41829121	52.88969654	66.89597728
	0.025	27.50341639	25.94585264	24.56893211	23.33825561	30.29427181
	0.050	14.51714039	13.66250439	12.90998692	12.23993223	16.0568046
	0.100	7.667982244	7.208446946	6.804820954	6.446376029	8.498012692
	0.250	3.482624435	3.279985683	3.102726461	2.946075936	3.849774279
	0.500	2.122980254	2.01370871	1.918774969	1.835538708	2.322185838
	1.000	1.492216472	1.43421367	1.384430089	1.341369114	1.599268701
	AEQL	0.264394363	0.252603169	0.242407716	0.233514582	0.285999763
	PCI	1.132239199	1.081744732	1.038083849	1	1.2247619
	RMI	0.144727436	0.090399477	0.042560489	0	0.242358726

The higher performance of the extended EWMA chart is consistent with the study of Karoon et al [21] which reported that when smoothing parameter $\lambda_2$is increasing. In addition, the findings that the adjusted EWMA-type are consistent show more effective in detecting changes the classical EWMA chart with previously presented such as studies showing in previous studies [24, 25].

According to the results from Tables 3 to 5 illustrate the study of the extended EWMA chart and original EWMA when choosing $\lambda_1 = 0.15$ and $\lambda_2 = 0.9\lambda_1$ to evaluated the ARL and overall performance criteria on ARX(3,1) model with varying the values of LCL from 0 to 0.075. The results insist that the extended EWMA control charts are more effective than the classical EWMA control chart for every different value of a, furthermore, the AEQL value indicate that the control charts are more slightly sensitivity when the LCL value increase.

Table 4. The ARL for the extended EWMA control chart on ARX(2,2) model compare with the EWMA chart given $\boldsymbol{\phi_1 = \phi_2 = 0.3, \beta_1 = 0.5}$ and $\boldsymbol{\beta_2 = 0.25}$

$\boldsymbol{\lambda_1}$	$\boldsymbol{\delta}$	$\boldsymbol{\lambda_2}$				**EWMA**
		$\boldsymbol{0.3\lambda_1}$ **(b =0.000469435)**	$\boldsymbol{0.5\lambda_1}$ **(b =0.000172602)**	$\boldsymbol{0.7\lambda_1}$ **(b =0.00006347)**	$\boldsymbol{0.9\lambda_1}$ **(b =0.00002334)**	**(b'=0.00210744)**
0.05	0.000	370.0623806	370.0851232	370.2705782	370.303918	370.0557168
	0.005	143.6099899	131.3741982	121.3469874	112.9932666	167.4925777
	0.010	88.1180071	78.84564694	71.50919257	65.57709639	107.3486736
	0.025	39.62228502	34.65638219	30.84069842	27.82653203	50.49432756
	0.050	19.70209443	16.97689337	14.90780681	13.28804074	25.81118976
	0.100	9.109697307	7.729595114	6.692445983	5.887764573	12.25996381
	0.250	3.095444374	2.614765165	2.265882038	2.005240113	4.241481434
	0.500	1.582397851	1.404972604	1.285218258	1.202671002	2.041950164
	1.000	1.132053547	1.079399244	1.048080721	1.029242898	1.289990658
	AEQL	0.210959608	0.194070285	0.182739658	0.174901114	0.255464393
	PCI	1.206165032	1.109600048	1.044817003	1	1.460621874
	RMI	0.336424167	0.191309977	0.083133016	0	0.669446511

0.10		(b =0.000949982)	(b =0.0003493386)	(b =0.0001284868)	(b =0.00004726)	(b'=0.004266463)
	0.000	370.0087502	370.0120179	370.0252132	370.0816262	370.0300218
	0.005	71.649121	63.8472325	57.75480129	52.88951613	88.24019188
	0.010	39.57608587	34.83510149	31.20579234	28.35098335	50.02999584
	0.025	16.82029518	14.65876241	13.02550689	11.75257144	21.71070307
	0.050	8.564602417	7.429641125	6.5759909	5.912616074	11.15848394
	0.100	4.361126688	3.781726754	3.348530314	3.013730289	5.698686323
	0.250	1.957676502	1.73723013	1.577566986	1.458460132	2.48618116
	0.500	1.303114058	1.210685405	1.148358179	1.105417891	1.543269823
	1.000	1.08108828	1.048751844	1.029524129	1.01795948	1.178254014
	AEQL	0.178945097	0.170070318	0.164204698	0.160209916	0.202791007
	PCI	1.116941452	1.061546765	1.024934671	1	1.265783113
	RMI	0.29562144	0.167910647	0.072906246	0	0.590748109
0.15		(b =0.0014279593)	(b =0.0005250648)	(b =0.0001931210)	(b =0.0000710357)	(b'=0.006418365)
	0.000	370.0007716	370.0137482	370.2646943	370.0054564	370.0056041
	0.005	54.72707532	48.48968749	43.67986131	39.86641746	68.30247889
	0.010	29.6429765	26.02645315	23.27907858	21.12757265	37.74195095
	0.025	12.59578915	10.99351383	9.787432007	8.849442004	16.25336403
	0.050	6.55216269	5.714438353	5.085790438	4.597862738	8.478559573
	0.100	3.494654209	3.063447519	2.741505761	2.492863279	4.494553824
	0.250	1.735702035	1.566190128	1.443509752	1.352019619	2.143298599
	0.500	1.243300745	1.169081181	1.119055437	1.084595034	1.436557432
	1.000	1.068790317	1.04135321	1.025043217	1.015233965	1.151365586
	AEQL	0.172403186	0.165235199	0.160521447	0.157327506	0.191804795
	PCI	1.095823545	1.050262621	1.020301219	1	1.219143424
	RMI	0.27876938	0.158247137	0.068716666	0	0.558574232

Table 5. The ARL of the EWMA and the extended EWMA control charts on ARX(3,1) model when the lower control limits ($\boldsymbol{a}$) are varied and $\boldsymbol{\lambda_1 = 0.15}$, $\boldsymbol{\lambda_2 = 0.9\lambda_1}$, $\boldsymbol{\phi_1 = 0.1}$, $\boldsymbol{\phi_2 = \phi_3 = 0.2}$, $\boldsymbol{\beta_1 = 0.25}$ are given

$\boldsymbol{a}$	Control chart	$\boldsymbol{\delta}$									AEQL	PCI	RMI
		0	**0.005**	**0.01**	**0.025**	**0.050**	**0.100**	**0.250**	**0.500**	**1.000**			
0	Extended b =0.00125924	370	56.45	30.62	12.99	6.729	3.561	1.744	1.241	1.066	0.200	1	0
	EWMA b'=0.1205195	370	134.9	82.73	38.55	20.65	10.975	4.932	2.904	1.912	0.407	2.034	1.403
0.025	Extended b =0.026263382	370	49.25	26.52	11.28	5.918	3.211	1.654	1.217	1.061	0.197	1	0
	EWMA b'=0.1488308	370	124.4	75.052	34.61	18.55	9.938	4.575	2.766	1.867	0.392	1.986	1.433
0.05	Extended b =0.051267388	370	42.84	22.94	9.807	5.220	2.909	1.574	1.194	1.057	0.195	1	0
	EWMA b'=0.17723	370	114.3	67.93	31.04	16.66	9.007	4.251	2.638	1.825	0.378	1.940	1.462
0.075	Extended b =0.07627128	370	37.19	19.84	8.539	4.619	2.647	1.504	1.175	1.052	0.192	1	0
	EWMA b'=0.2057207	370	104.7	61.37	27.83	14.97	8.173	3.956	2.519	1.785	0.364	1.894	1.489

4.3. Applications

Typically, data in economics and financial applications is collected over specific periods, such as daily, weekly, or monthly. Current observations often depend on previous data, leading to correlations within the data itself. In this context, the SCB stock price (measured in THB) and the exchange rate (USD/THB) are considered exogenous variables. Data was collected daily from January 4, 2022, to June 28, 2022. The second application examines the Thailand GDP percentage expansion (%YoY) incorporating two exogenous variables: the exports (%) and the imports (%). This data was collected quarterly from 2001 to 2020 and is of great interest for study. In this section, the explicit formula for evaluating the ARL of the ARX(p,r) model on the extended EWMA control chart are applied to the real datasets. We also compare the efficiency of detecting process changes with the traditional control chart.

The two applications, SCB stock price and GDP percentage expansion are tested for autocorrelation using the Box-Jenkins time series technique. The exogenous variables also incorporate to the prediction models for test the significant effect in the model. Table 6 reveals t-test statistic, the coefficient estimation and the Root Mean Square Error (RMSE) and Normalized Bayesian Information Criterion (BIC) values from the fitted ARX(p,r) model. The result indicates that ARX(1,1) model is the most suitable model for describing the pattern of the first application due to the lowest RMSE and Normalized BIC values. For GDP percentage expansion observation, the prediction model ARX(1,2) has the lowest of RMSE and normalized BIC values.

To apply the explicit formula to the practical situation, the residuals between the actual values and the prediction assumed as the exponential white noise are determined by Kolmogonov-Smirnov testing. Table 7 shows that the residuals of the optimal model of two applications are all exponentially distributed. Therefore, the prediction model, ARX(1,1) for the SCB stock price $(Y_t)$with exchange rate (USD/THB) as input variable (X_{1t})can be assigned as

$$Y_t = 0.956Y_{t-1} + 3.432X_{1t} + \varepsilon_t$$

where the in-control parameter, α_0= 2.334.

For the second dataset, the residuals follow the exponential distribution, $\varepsilon_t \sim Exp(1.426)$ when the process is an in-control state. Thus, the prediction model ARX(1,2) for GDP percentage expansion $(Y_t)$with the export $(X_{1t})$and import (X_{2t}) can be written as

$$Y_t = 0.819Y_{t-1} + 0.255X_{1t} + 0.062X_{2t} + \varepsilon_t$$

Tables 8 and 9 demonstrate the ARL values calculated form the explicit formula of ARX(1,1) and ARX(1,2) for the two real-word data when the smoothing parameter λ_1= 0.05, 0.10, 0.15 and λ_2=$0.3\lambda_1$,$0.5\lambda_1$,$0.7\lambda_1$,$0.9\lambda_1$was set. The results confirm that the extended EWMA control chart quicklier detecting changes than the EWMA control chart and also show the better performance when $\lambda_2$increased and close toλ_1, The overall performance measure AEQL, PCI and RMI also present in Figures 1 and 2 for stock price and GDP, respectively

Table 6. The ARX(p,r,) estimation and the model fit for applications

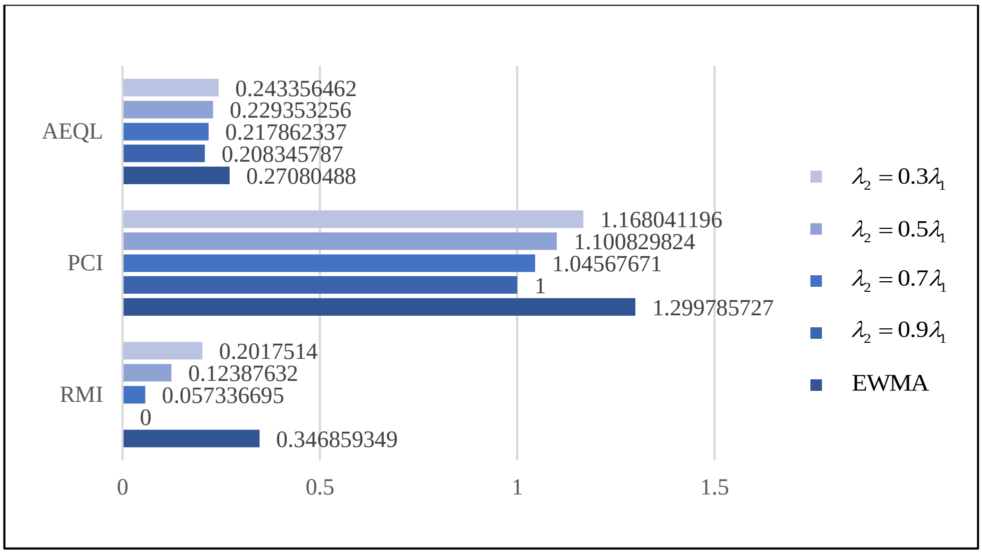
Data	Model	Variables	Coefficient	Std.	t	Sig	Model fit	
							RMSE	Normalized BIC
SCB stock price	ARX(1,1)	AR(1) $(\hat{\phi}_1)$	0.956	0.029	33.041	0.000	**3.915**	**2.813**
		Exchange rate$(\hat{\beta}_1)$	3.432	0.207	16.566	0.000		
	ARX(2,1)	AR(1) $(\hat{\phi}_1)$	0.867	0.095	9.133	0.000	3.930	2.863
		AR(2) $(\hat{\phi}_1)$	0.094	0.096	0.979	0.330		
		Exchange rate$(\hat{\beta}_1)$	3.419	0.233	14.650	0.000		
GDP percentage expansion	ARX(1,2)	AR(1) $(\hat{\phi}_1)$	0.819	0.067	12.238	0.000	**1.807**	**1.347**
		Export$(\hat{\beta}_1)$	0.255	0.037	6.806	0.000		
		Import$(\hat{\beta}_2)$	0.062	0.030	2.075	0.041		
	ARX(2,2)	AR(1) $(\hat{\phi}_1)$	0.902	0.116	7.793	0.000	1.811	1.407
		AR(2) $(\hat{\phi}_1)$	-0.103	0.115	-0.901	0.370		
		Export$(\hat{\beta}_1)$	0.254	0.036	7.044	0.000		
		Import$(\hat{\beta}_2)$	0.072	0.029	2.468	0.016		

Table 7. Exponential white noise testing

Data	Model	Mean (α_0)	Kolmogorov-Smirnov Z	Sig.
SCB stock price	ARX(1,1)	2.334	0.854	**0.460**
GDP percentage expansion	ARX(1,2)	1.426	0.917	**0.370**

Table 8. The ARL for ARX(1,1) applying to SCB stock price data

λ_1	δ	λ_2				EWMA (b'=0.0317708)
		$0.3\lambda_1$ (b =0.018917715)	$0.5\lambda_1$ (b =0.013406835)	$0.7\lambda_1$ (b =0.0095070806)	$0.9\lambda_1$ (b =0.00674426)	
0.05	0.000	370.0040501	370.0007067	370.0004582	370.0599657	370.3089563
	0.005	106.820121	100.4375606	94.729478	89.60972967	117.9773533
	0.010	62.42901753	58.09904009	54.29967473	50.94787977	70.20205248
	0.025	27.81192489	25.66414057	23.80794348	22.19160901	31.75271609
	0.050	14.49887529	13.32686357	12.32088143	11.4500065	16.67061524
	0.100	7.489519175	6.867501306	6.336730702	5.879612506	8.65150742
	0.250	3.257345392	2.993835405	2.771643106	2.582446113	3.757137073
	0.500	1.935603515	1.801737894	1.690857261	1.598142672	2.195037645
	1.000	1.365284958	1.300277016	1.247971345	1.205547605	1.495656746
	AEQL	0.243356	0.229353	0.217862	0.208346	0.270805
	PCI	1.168041	1.10083	1.045677	1	1.299786
	RMI	0.201751	0.123876	0.057337	0	0.346859
0.10		b = 0.038019014	b = 0.0269074	b = 0.019065803	b = 0.013519388	b'= 0.0641
	0.000	370.0008847	370.0180957	370.002168	370.0111296	370.1931347
	0.005	78.12655773	72.89852825	68.31871872	64.27717034	87.58161942
	0.010	43.89350325	40.63849202	37.82925685	35.38170207	49.90909725
	0.025	19.21316807	17.69225938	16.39399277	15.2734093	22.06984454
	0.050	10.14435448	9.327202978	8.633014535	8.036299121	11.68975246
	0.100	5.448509626	5.013056812	4.644729563	4.329359697	6.276711769
	0.250	2.614675638	2.423520999	2.263449008	2.127751184	2.982606894
	0.500	1.707551758	1.605294058	1.521063555	1.45087754	1.908159639
	1.000	1.296473483	1.243334201	1.200767108	1.166335662	1.404112067
	AEQL	0.220553	0.209613	0.200696	0.193346	0.242281
	PCI	1.140715	1.084136	1.038015	1	1.253096
	RMI	0.194682	0.119009	0.05491	0	0.338152
0.15		b = 0.057220513	b = 0.04044011	b = 0.02863122	b = 0.02029285	b'= 0.096866815
	0.000	370.0002681	370.00073	370.0373006	370.0491258	370.0012379
	0.005	70.05726393	65.15864916	60.92132591	57.63997126	79.10781293
	0.010	38.96391285	35.98501512	33.44259984	31.24396259	44.5831598
	0.025	17.01228401	15.64200086	14.48366842	13.49029437	19.6358306
	0.050	9.040053548	8.306713536	7.689271449	7.161623511	10.45229959
	0.100	4.928824771	4.537551996	4.209261916	3.929640297	5.685608497
	0.250	2.446797359	2.273286471	2.128957938	2.007123417	2.785598961
	0.500	1.64587503	1.551618886	1.474405078	1.410285066	1.833044172
	1.000	1.277067435	1.227051647	1.187164634	1.15498856	1.379405859
	AEQL	0.214403	0.204243	0.196007	0.189245	0.234829
	PCI	1.132942	1.079251	1.035732	1	1.240876
	RMI	0.192439	0.116833	0.05331	0	0.338134



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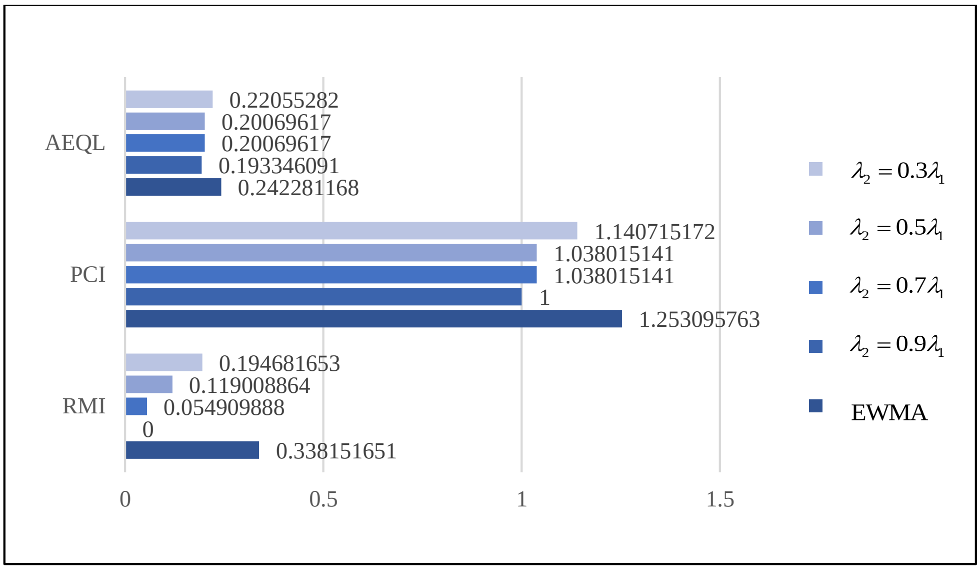

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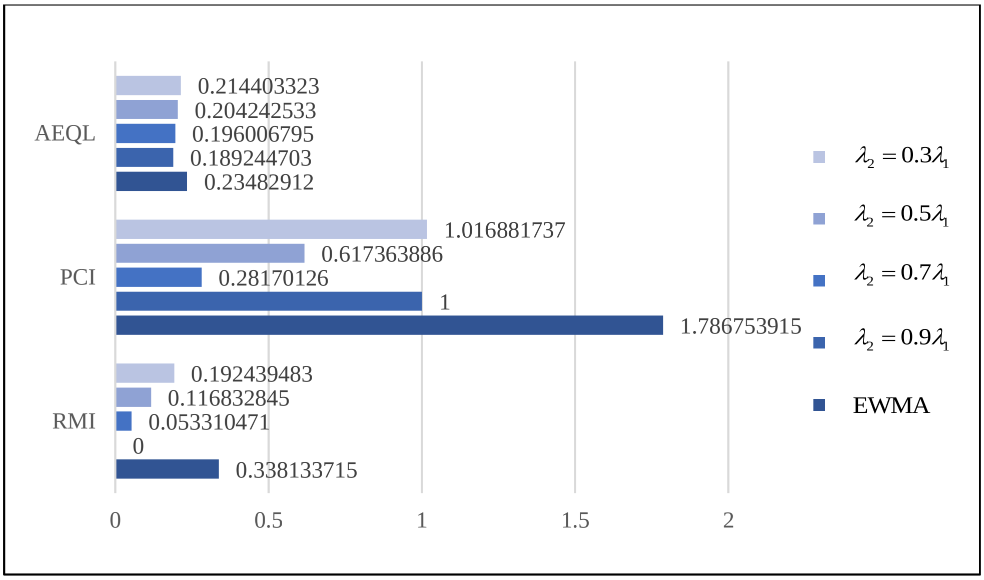

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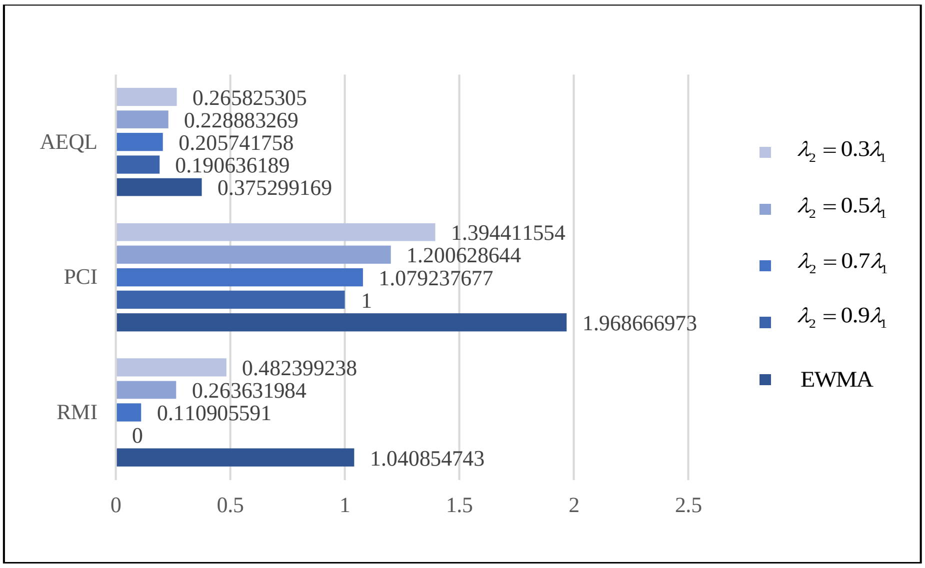
Figure 1. AEQL, PCI and RMI value on the control charts for SCB stock price application where (a) $\lambda_1 = 0.05$, (b) $\lambda_1 = 0.10$ and (c) $\lambda_1 = 0.15$

Table 9. The ARL for ARX(1,2) applying to GDP percentage expansions data

λ_1	δ	λ_2				EWMA (b'=0.01069878)
		$0.3\lambda_1$ (b =0.00229284)	$0.5\lambda_1$ (b =0.000823678)	$0.7\lambda_1$ (b =0.000296076)	$0.9\lambda_1$ (b =0.00010644541)	
0.05	0.000	370.0195125	370.0168014	370.0111423	370.0010579	370.0193069
	0.005	176.1708882	158.5753877	144.2945208	132.6327563	210.3047398
	0.010	114.6481409	99.88600383	88.54410867	79.67349984	146.0956492
	0.025	54.67975265	46.01742504	39.69272597	34.92894616	74.99295654
	0.050	28.07796619	23.1364354	19.61045931	16.99686804	40.2267433
	0.100	13.32666035	10.75927865	8.958461228	7.640384668	19.86256417
	0.250	4.553212746	3.613581107	2.977569056	2.528109829	7.079400055
	0.500	2.142852181	1.762440126	1.520724827	1.361474675	3.247087222
	1.000	1.318633449	1.18534282	1.10951606	1.065340391	1.757527932
	AEQL	0.265825	0.228883	0.205742	0.190636	0.375299
	PCI	1.394412	1.200629	1.079238	1	1.968667
	RMI	0.482399	0.263632	0.110906	0	1.040855
0.10		b =0.00464517	b =0.0016683882	b =0.0005998031	b =0.0002156943	b'=0.021751807
	0.000	370.0844572	370.004395	370.0081256	370.0006901	370.0002399
	0.005	93.258375	80.48276256	70.90113885	63.53864909	121.8164902
	0.010	53.2575307	45.03837717	39.07747821	34.60957764	72.87038299
	0.025	23.20433801	19.30221432	16.54093729	14.50635031	33.00876762
	0.050	11.91624534	9.836255509	8.379050339	7.312193249	17.26214393
	0.100	6.056588517	4.981442311	4.235229765	3.692728425	8.873488972
	0.250	2.602744369	2.175942576	1.888509094	1.685962798	3.769792565
	0.500	1.587676052	1.391521285	1.267234871	1.18545813	2.163745959
	1.000	1.193841436	1.112669882	1.066563681	1.039715824	1.463351631
	AEQL	0.207328	0.187726	0.175636	0.167864	0.266973
	PCI	1.235092	1.11832	1.046298	1	1.590409
	RMI	0.434137	0.236377	0.09922	0	0.947839
0.15		b =0.006987309	b =0.002508225	b =0.00090163	b = 0.0003242357	b'=0.032870324
	0.000	370.0015582	370.0069603	370.0544269	370.0026892	370.0008461
	0.005	72.02856861	61.51285853	53.79130086	47.9445093	96.72326424
	0.010	39.99670149	33.62557532	29.07021455	25.68715058	55.77959201
	0.025	17.25654106	14.34934962	12.30871606	10.81213801	24.75080983
	0.050	8.984280125	7.44793	6.377499842	5.596034753	13.01337024
	0.100	4.736081026	3.937189241	3.384884769	2.984071306	6.863927909
	0.250	2.225599379	1.898323953	1.678483015	1.523718346	3.131910337
	0.500	1.469854915	1.312777862	1.213434874	1.148109584	1.935891627
	1.000	1.163970193	1.095243656	1.056258866	1.033566465	1.39405184
	AEQL	0.195216	0.179327	0.169584	0.163354	0.244267
	PCI	1.19505	1.097785	1.038144	1	1.49533
	RMI	0.412786	0.224275	0.094051	0	0.910455



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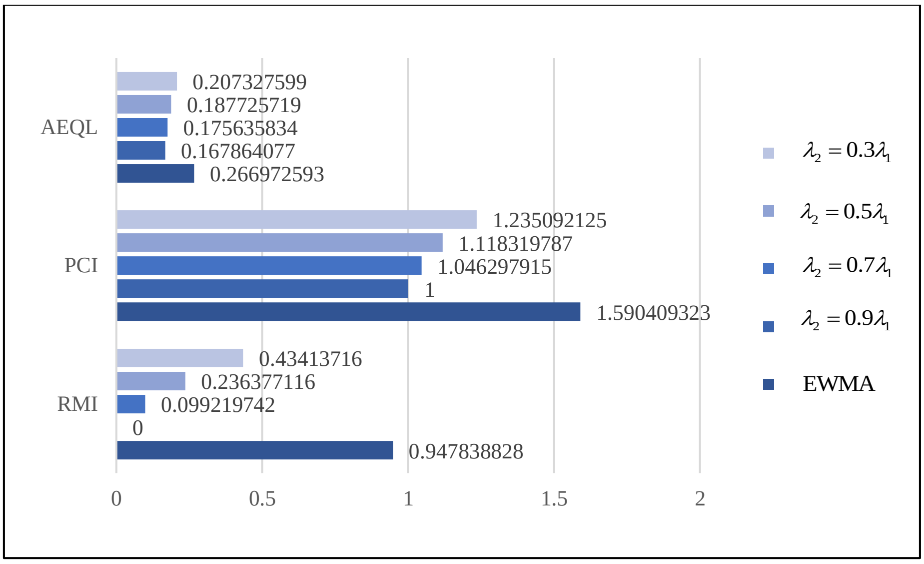

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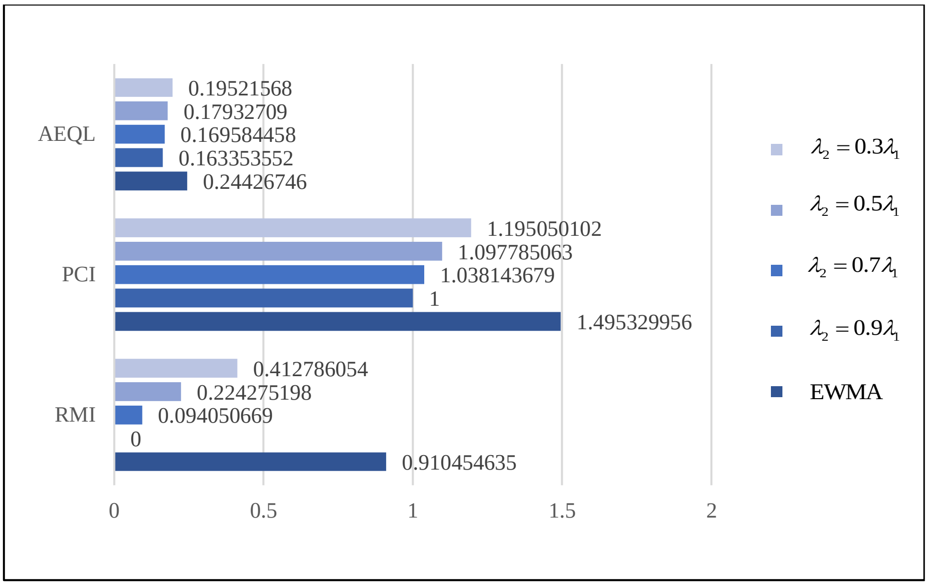

(c)

Figure 2. AEQL, PCI and RMI value on the control charts for GDP percentage expansions application where (a) $\lambda_1 = 0.05$, (b) $\lambda_1 = 0.10$ and (c) $\lambda_1 = 0.15$

5. Conclusion

The capacity of the control charts in capturing changes in the process is usually assessed by the general characteristic, the ARL, which can be described through the Fredholm integral equation of the second kind. This research focused on the alternative methodology in calculating the ARL of the extended EWMA control chart for the ARX model with exponential white noise. The explicit formula derived from the ARL integral equation is proposed and proved the existence and uniqueness by applying the condition of Banach's fixed point theorem. The accuracy of the exact solutions is verified by NIE methods with four different composite quadrature rules. The result indicates that the ARL from two methods is close, and the computation time of the proposed explicit formulas is less than 0.001 second.

The second purpose of this study is to compare the sensitivity of the extended EWMA and the classical EWMA control charts under various situations and also examine the optimal condition of the smoothing parameter of the EWMA-type charts. It can be seen that the extended EWMA control chart shows better performance in detecting process mean changes, especially small shift sizes, as confirmed by overall performance criteria such as AEQL, PCI, and RMI values. Moreover, the result indicated that the extended EWMA control chart has higher efficiency when the smoothing parameter $\boldsymbol{\lambda_2}$ is almost equal to $\boldsymbol{\lambda_1}$. The two real datasets, namely SCB stock price and GDP percentage expansions with external factors, are applied to demonstrate the performance of the relevant control charts when the residuals of the forecasting model are exponentially distributed.

However, the proposed procedure, an explicit formula, works in some conditions, in particular, when data is autocorrelated with exponential white noise. For future study, an explicit formula for the ARL will be developed for other time series models running on extended EWMA or the new adjusted control charts.

6. Declarations

6.1. Author Contributions

Conceptualization, T.M., Y.A., and S.S.; methodology, T.M.; software, T.M.; validation, T.M., Y.A., and S.S.; formal analysis, T.M.; investigation, Y.A.; resources, T.M.; data curation, Y.A.; writing—original draft preparation, T.M.; writing—review and editing, T.M.; visualization, Y.A.; supervision, Y.A.; project administration, T.M.; funding acquisition, Y.A. All authors have read and agreed to the published version of the manuscript.

6.2. Data Availability Statement

The SCB stock price and the GDB percentage expansions datasets can be found here: *https://investing.com* and *https://www.nesdc.go.th*, respectively.

6.3. Funding

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6.4. Institutional Review Board Statement

Not applicable.

6.5. Informed Consent Statement

Not applicable.

6.6. Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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