Available online at www.HighTechJournal.org

HighTech and Innovation Journal

ISSN: 2723-9535

Vol. 6, No. 3, September, 2025

Exact Run Length Sensitivity of DEWMA Control Chart Based on Quadratic Trend Autoregressive Model

Yupaporn Areepong [1], Kotchaporn Karoon [2*]

[1] *Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.*

[2] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand.*

Received 13 April 2025; Revised 15 July 2025; Accepted 03 August 2025; Published 01 September 2025

Abstract

One well-known process detection tool that is sensitive to even little shift changes in the process is the Double Exponentially Weighted Moving Average (DEWMA) control chart. The present study aims to provide exact average run length (*ARL*) on the DEWMA chart under the data that is underlying the quadratic trend autoregressive (AR) model. At that point, the computed *ARL* via the numerical integral equation (NIE) technique was compared in terms of accuracy to the exact one that was developed by using the percentage accuracy (%*Acc*). And then, the computational times of both were also compared. The results revealed that the ARL results of exact *ARL* and *ARL* via the NIE method show hardly any difference in terms of accuracy, but exact *ARL* outperformed in terms of computational times that were computed instantly, whereas the other way spent approximately 2-3 seconds computing. Thereafter, the proposed *ARL* operating on the DEWMA chart was compared to the CUSUM and EEWMA charts. It was found to be more effective in terms of detection performance. Especially when there are little shift changes in the process. The run length formulas, which are the standard deviation run length (*SDRL*) and the median run length (*MRL)*, were measures of sensitivity evaluation and were used to verify their capability. The sensitivity of detecting changes of exact *ARL* running on the DEWMA chart was illustrated by the real data utilized in fields of economics about natural gas importing in Thailand (Unit: 100 MMSCFD at heat value of natural gas 1,000 BTU/SCF). Apparently, the exact *ARL* of the DEWMA chart is an excellent choice to detect small shift changes under this scenario, which represents properties as a quadratic trend AR model.

Keywords: Autoregressive Model; DEWMA Control Chart; Exact Run Length; Explicit Formula; Quadratic Trend.

1. Introduction

The control chart is a statistical tool that is frequently used to detect process changes and monitor the quality of manufacturing processes. The Shewhart control chart is the most used because of its simplicity and high sensitivity to major changes in the process. However, it is not great for detecting minor to moderate changes in the process; hence, researchers have developed other control charts for each scenario. Two widespread instances are the cumulative sum (CUSUM) [1] and exponentially weighted moving average (EWMA) [2] control charts. Many studies also noted control charts that were modified from the EWMA-type chart. They were more sensitive than the standard EWMA chart in detecting even minute changes in the process. Examples of control charts include the modified exponentially weighted moving average (MEWMA) [3] and the extended exponentially weighted moving average (EEWMA) [4]. Many researchers have studied those control charts and noticed that they are sensitive enough to detect tiny changes in the process for different scenarios. Moreover, Shamma and Shamma first introduced the double exponentially weighted

* Corresponding author: kotchapornk@nu.ac.th

http://dx.doi.org/10.28991/HIJ-2025-06-03-07

moving average (DEWMA) [5]. After that, Mahmoud and Woodall adjusted it [6] and demonstrated that the DEWMA chart is an alternative for more sensitivity for detecting tiny changes in the process parameters. It has widespread application in a variety of fields and procedures, including finance, economics, medicine, healthcare, and the environment.

Autocorrelation describes the tendency of subsequent points of data in a time series to correspond. Control charts are statistical techniques used to find out when behavior becomes out of control. One technique for dealing with autocorrelation in a control chart is to employ specific algorithms that consider the correlation between subsequent data points. For example, a control chart for the autoregressive and moving averages (ARMA) model detects a process with autocorrelated data by combining time series models with control charts. On top of that, the trend and quadratic trend are two aspects of autocorrelation that might impact the dependent variable, such as indicators for forecast data in various fields. Most real-world data takes the form of time series with either linear trends or quadratic trends as components. Karaoglan & Bayhan [7] applied a trend-stationary AR(1) model to estimate the peroxide amounts in stored vegetable oil. Yue & Pilon [8] investigated an annual mean daily streamflow dataset from 15 watersheds, utilizing a linear trend with the AR(1) model. Karoon et al. [9] applied a quadratic trend AR(p) model and a control chart combination to monitor user web browser data in Thailand.

The average run length (*ARL*) is the most frequently implemented metric to quantify control chart efficacy in the process. There exist two properties: in-control *ARL* (ARL_0) and out-of-control *ARL* (ARL_1). ARL_0 represents the average number of observations a process in control renders before signalling that it is out of control, and it should be high. To identify an out-of-control adjustment in a process variable, an average number of observations, known as ARL_1, is necessary, and it must be as few as possible. *ARL* computations have been made to use a variety of methods, as suggested by various literary works. comprised of Monte Carlo simulation, Markov chain, and the numerical integral equation (NIE). They were found in Champ & Rigdon [10], Riaz et al. [11], and Peerajit [12]. Furthermore, some scholars employ and advocate the computation of these indicators as well as the calculation of run length (RL) using the measures of central tendency (median) and spread (standard deviation), which can be called *MRL* and *SDRL*, respectively. They were extra measurements used to track shift changes in the process.

In time series analysis, it is crucial to consider the error, which is the difference between the observed and predicted values. A smaller error generally indicates higher model accuracy. This error, commonly referred to as white noise, is typically assumed to follow a normal distribution. However, in cases where the data exhibit autocorrelation, the error structure may instead follow an exponential white noise pattern. To assess the effectiveness of control charts, specific formulas are used. One of these formulas is derived from the Fredholm integral equation of the second kind, which requires the proof of the existence and uniqueness of the *ARL* using Banach's fixed-point theorem in order to arrive at a complete formula. Many researchers have extended this approach, originally developed for *ARL*, to other control charts and diverse applications. Starting with Supharakonsakun [13], a two-sided exact *ARL* formula for the modified EWMA chart was created using the generic moving average (MA(p)) model, and it was then applied to the Dow Jones composite average based on a real-life dataset.

Bualuang & Peerajit [14] demonstrated explicit and NIE of *ARL* operating on the CUSUM chart, as well as the ARFIX process, and applied it to an economic dataset containing gold futures prices. Karoon & Areepong [15] recently published an explicit *ARL* based on the general AR with the trend model of the double EWMA chart and applied it to economic data containing cryptocurrency prices. Phanyaem [16] proposed explicit solutions for the ARL of the exponentially Weighted Moving Average (EWMA) control chart in the presence of a $SARX(P,r)_L$ process. In the same year, Phanyaem [17] provide formulas for computing the ARL of the EWMA chart for quadratic trend AR(1) model with exponential white noise. In the literature mentioned above, it was demonstrated that the capability of an exact *ARL* solution with any control chart under autocorrelated data can be applied to current real-life data. Moreover, Karoon & Areepong [18] presented the exact *ARL* solution for the new EEWMA control chart under the AR model and compared the performance of this new EEWMA chart with the traditional EWMA and extended EWMA charts. The comparison was also applied to an economic dataset from Thailand. Recently, Neammai et al. [19] used the MA(q) process to create an analytical formula for the ARL of DMEWMA charts. Their findings indicate that, for various process mean shifts, the DMEWMA chart outperforms others, with stock data demonstrating its superior efficacy in process monitoring.

According to the literature review, the quadratic trend component in general AR models has been incorporated into various control charts, such as the extended EWMA [9] and the Adjusted modified EWMA [20] in 2023 and 2024, respectively. However, no application has been reported for the DEWMA chart based on quadratic trend AR(p) model. In 2025, this study, therefore, presents the *ARL* of the DEWMA chart for general AR models using the quadratic trend model, also known as the quadratic trend AR(p) model. The calculation was performed using two methods: the exact solution and the NIE technique. Additionally, the defined *ARL* had not been previously addressed. A comparison of both methods was made in terms of accuracy and computation speed under the two-sided DEWMA chart. Both simulated and real-world economic data were then compared with the EEWMA and CUSUM charts, as well as the accurate *ARL* DEWMA chart. Moreover, real-life data is used in this study to demonstrate the capability of DEWMA control charts. It was also verified by detecting changes in control charts by showing a graph-quality control chart.

2. Preliminaries

This section presents a brief description of the two-sided control charts and the quadratic trend AR(p) model with exponential white noise.

2.1. Structure of Control Chart

First, Page [1] created the CUSUM chart for quality control, which can be used as a substitute for the Shewhart control chart to identify slight to moderate shift changes in the process. The CUSUM chart's statistics can be stated characteristically in Equation 1 as follows:

$$C_t = max(0, C_{t-1} + X_t - \kappa); \quad t = 1\,.\,2, ... \tag{1}$$

where X_t is a sequence of quadratic trend AR(p) process with an exponential white noise.

$C_0 \geq 0$ and $\kappa > 0$ are the starting value and the non-zero constant, respectively, and then $C_0 = \varpi$ is the initial value of CUSUM; $\varpi \in [a, b]$. Moreover, $\kappa > 0$ is signalled that the process may be out-of-control. The CUSUM showed the corresponding stopping time as $\tau_C = inf\{t \geq 0;\ C_t < LCL \ \ or \ \ C_t > UCL\}$ where a and b are expressed as the lower (LCL) and upper (UCL) control limit of two-sided CUSUM chart.

Second, the Extended Exponentially Weighted Moving Average (EEWMA) control chart was introduced by Naveed et al. [4] after Roberts [2] developed the EWMA chart to monitor a process over time. It works well for tracking and identifying slight variations in the average procedure. It is possible to express the EEWMA control chart using the recursive equation in Equation 2.

$$EE_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1 - \lambda_1 + \lambda_2)E_{t-1},; \ \ t = 1, 2, ... \tag{2}$$

where λ_1 and λ_2 are exponential smoothing parameters with interval as $(0 < \lambda_1 \leq 1)$ and $(0 < \lambda_2 < \lambda_1)$, respectively. The initial value is a constant, $EE_0 = u$. On the EEWMA chart, the upper and lower control limits are provided by:

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}, \tag{3}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}, \tag{4}$$

where μ_0, σ, and L are the mean, the process standard deviation, and the suitable control limit width, respectively.

he mean and variance of the process variable X_t, which is monitored by the EEWMA statistic, are denoted by $\mu_0$and $\left(\sigma^2\left[\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}\right]\right)$, respectively. The EEWMA control chart's stopping time can be found using $\tau_{EE} = inf\{t \geq 0; EE_t < LCL \quad or \quad EE_t > UCL\}$ where c and $d$are expressed as the lower (LCL) and upper (UCL) control limit of two-sided EEWMA chart.

Third, Shamma & Shamma [5] updated the classic EWMA chart to create the DEWMA chart, which was later created by Mahmoud & Woodall [6] in 2010 to efficiently monitor tiny changes in process parameters. The DEWMA chart's statistics can be estimated using Equation 5:

$$E_t = \lambda_2 X_t + (1 - \lambda_2)E_{t-1}\,; \ \ t = 1, 2, ... \ \text{ and } DE_t = \lambda_1 E_t + (1 - \lambda_1)DE_{t-1} \tag{5}$$

where λ_1 and λ_2 are exponential smoothing parameters with intervals that are $(0 < \lambda_1 \leq 1)$ and $(0 < \lambda_2 < 1)$, respectively, the apparent exponential smoothing parameters of the EEWMA chart. And then, DE_t with $t = 0$ represented the initial value of the DEWMA statistics, $DE_0 = v$. The upper (UCL) and lower (LCL) control limits of the DEWMA chart are as follows:

$$UCL = \mu_0 + \ddot{L}\sigma\sqrt{\frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\left[\frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2} + \frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2} - 2\frac{(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right]},, \text{ and} \tag{6}$$

$$UCL = \mu_0 - \ddot{L}\sigma\sqrt{\frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\left[\frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2} + \frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2} - 2\frac{(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right]}, \tag{7}$$

where μ_0, σ , and $\ddot{L}$ are the mean, the process standard deviation, and the suitable control limit width, respectively.

The process variable X_t used in constructing the DEWMA statistics has mean (μ_0) and variance $\left(\frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\sigma^2\left[\frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2} + \frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2} - 2\frac{(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right]\right)$, respectively. The EEWMA control chart's stopping time can be found using $\tau_{DE} = inf\{t \geq 0; DE_t < LCL \quad or \quad DE_t > UCL\}$ where e and $f$are expressed as the lower (LCL) and upper (UCL) control limit of two-sided DEWMA chart. Moreover, both the DEWMA and EEWMA statistics are equivariant, as they can be transformed into the traditional EWMA statistic. Specifically, when $\lambda_2$in the DEWMA statistic is set to 1, the DEWMA statistic reduces to the standard EWMA statistic. Similarly, when $\lambda_2$in the EEWMA statistic is set to 0, the EEWMA statistic becomes equivalent to the EWMA statistic.

2.2. Methodology for Exact ARL on DEWMA Chart based on Quadratic Trend AR Model

Primarily, this study derives the exact formulas of the Quadratic Trend AR(p) model. The quadratic trend AR(p) model is a statistical model used to analyse and forecast time-series data, which is data in which observations are collected over time and their order is important. Modelling time-series data is frequently employed in many disciplines, such as economics, finance, engineering, and environmental research. This study focused on the general quadratic trend autoregressive model, often known as the quadratic trend AR(p) model. Equation 8 represents the quadratic trend AR(p) model for lag p.

$$X_t = \psi + \eta t + \vartheta t^2 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_p X_{t-p} + \xi_t \tag{8}$$

where ψ is the constant of model, η and ϑ are the constant of times, and both linear and quadratic trend time terms (t and t^2) were included as exogenous variables to capture the data's underlying trend. $\phi_1, \phi_2, \ldots, \phi_p$ are the coefficients of time series model with $|\phi_1, \phi_2, \ldots, \phi_p| < 1$. Also, $\xi_t$is the error term for continuous i.i.d. random variables derived from exponential white noise; $\xi_t \sim Exp(\gamma)$. The probability density function of ξ_t can be expressed as $f(x,\gamma) = \frac{1}{\gamma e^{-\frac{x}{\gamma}}}; \gamma > 0$. Based on the *ARL* characteristics considered throughout the study, several simple change-point models are analyzed as follows below:

$$\xi_t \sim \begin{cases} Exp(\gamma_0) & t = 1, 2, ..., \theta - 1 \\ Exp(\gamma_1) & t = \theta, ..., \theta + 1, ... \end{cases} \tag{9}$$

where γ_0 and γ_1 are known parameters with $\gamma_1 > \gamma_0$. By exploring the change point in Equation 5, the *ARL* defined with $E_\theta(.)$ can be described as follows below.

$$ARL = \begin{cases} ARL_0 = E_\infty(\tau), \theta = \infty & (no \quad change) \\ ARL_1 = E_1(\tau), \theta = 1 & (change) \end{cases} \tag{10}$$

where $E_\theta(.)$denotes the expectation under distribution $F(x,\gamma)$ for a given change-point time. $\theta = \infty$ shows in-control *ARL* (ARL_0), whereas $\theta = 1$ indicates the initial instance of a change from γ_0 to γ in the process, which is known as out-of-control *ARL* (ARL_1). Subsequently, the DEWMA statistic defined in Equation 5 can be reformulated using the quadratic trend AR(p) model, and is represented as follows:

$$DE_t = \lambda_1\lambda_2(\psi + \eta t + \vartheta t^2 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_p X_{t-p} + \xi_t) + \lambda_2(1-\lambda_1)E_{t-1} + (1-\lambda_2)DE_{t-1} \tag{11}$$

Under the in-control condition, the DEWMA scheme is defined as a two-sided control chart, with $e < DE_t < f$; Then,

$$e < \lambda_1\lambda_2(\psi + \eta t + \vartheta t^2 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \xi_t) + \lambda_2(1-\lambda_1)E_{t-1} + (1-\lambda_2)DE_{t-1} < f \tag{12}$$

Subsequently, the equation was rewritten in terms of ξ_t with the change-point time at $t = 1$, and initial values are defined as $DE_0 = v$ and $E_0 = z$. The interval of ξ_t can be rearranged as

$$\frac{e-(1-\lambda_1)v}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega < \xi_1 < \frac{f-(1-\lambda_1)v}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega \tag{13}$$

where ω represents $\psi + \eta + \vartheta + \sum_{i=1}^{p} \phi_i X_{1-i}$.

Let $\zeta(v)$ be the exact *ARL* on DEWMA chart under quadratic trend AR(p). The exact ARL in this study was modified using the Fredholm integral equation of the second kind [21], as presented below:

$$\zeta(v) = 1 + \int_{\frac{e-(1-\lambda_1)v}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega}^{\frac{f-(1-\lambda_1)v}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega} \zeta(\lambda_1\lambda_2(\omega + \xi_1) + (1-\lambda_1)v + \lambda_1(1-\lambda_2)z) g(\xi_1) d\xi_1 \tag{14}$$

Let ρ denotes $\lambda_1\lambda_2(\omega + \xi_1) + (1-\lambda_1)v + \lambda_1(1-\lambda_2)z$, then $\frac{d\rho}{d\xi_1} = \lambda_1\lambda_2$ and $d\xi_1 = \frac{1}{\lambda_1\lambda_2} = d\rho$. From Equation 15, the integral variable was changed; it can be rewritten as Equation 15:

$$\zeta(v) = 1 + \frac{1}{\lambda_1\lambda_2} \int_e^f \zeta(\rho) \cdot g\left(\frac{\rho-(1-\lambda_1)v}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega\right) d\rho \tag{15}$$

Here, ξ_t was determined as $\xi_t \sim Exp(\gamma)$. Thus, the exact *ARL* generated by the second-kind Fredholm integral equation can be shown as follows:

$$\zeta(v) = 1 + \frac{\Lambda(v) \cdot M}{\gamma\lambda_1\lambda_2} \tag{16}$$

where $\Lambda(v) = e^{\frac{(1-\lambda_1)v}{\gamma\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\gamma\lambda_2} - \frac{\lambda_1\lambda_2\omega}{\gamma}}$, $M = \int_e^f \zeta(\rho) M_0(\rho) d\rho$, $M_0(\rho) = e^{-\frac{\rho}{\gamma\lambda_1\lambda_2}}$. According to Equation 17, the following holds:

$$M = \int_e^f M_0(\rho)\left(1 + \frac{\Lambda(v) \cdot M}{\gamma\lambda_1\lambda_2}\right) d\rho = -\frac{\gamma\lambda_1\lambda_2[M_0(f) - M_0(e)]}{1 + \frac{1}{\lambda_1} e^{\frac{(1-\lambda_2)z}{\gamma\lambda_2} + \frac{\omega}{\gamma}} \cdot [M_0(f) - M_0(e)]} \tag{17}$$

Finally, from Equation 17, it can be rearranged as the exact ARL running on the two-sided DEWMA chart under the quadratic trend AR(p) model, and that is expressed in the form of Equation 18:

$$\zeta(v) = 1 - \frac{\lambda_1 e^{\frac{(1-\lambda_1)v}{\gamma\lambda_1\lambda_2}} \cdot [M_0(f) - M_0(e)]}{\lambda_1 e^{\frac{1}{\gamma}\left(\frac{(1-\lambda_2)z}{\lambda_2}+\omega\right)} + [M_0(\lambda_2 f) - M_0(\lambda_2 e)]} \tag{18}$$

Besides, replace $\gamma = \gamma_0$ in Equation 18 showing the in-control situation, $\gamma = \gamma_1 = \gamma_0(1+\delta)$ might depict the out-of-control situation.

In the next step, Numerical Integral Equation (NIE) Technique of the Quadratic Trend AR(p) model is derived. Let $\hat{\zeta}(v)$ be ARL of the DEWMA chart that is derived by NIE technique, to estimate the interval $[e, f]$ in terms of n linear equation systems, the Gauss-Legendre rule was applied, and it has been divided into $e \le r_1 \le \ldots \le r_n \le f$. The approximate formula for an integral is shown below.

$$\int_e^f \zeta(\rho) g(\rho) d\rho \approx \sum_{j=1}^{n} w_j g(r_j) \tag{19}$$

where $w_j = (f-e)/n$, and $r_j = (j-0.5)w_j + e$ with $j = 1,2, \ldots, n$.

Using the quadrature formula, the following result is obtained $\hat{\zeta}(r_i) = 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n} w_j \cdot \hat{\zeta}(r_j) \cdot g\left(\frac{r_j-(1-\lambda_1)r_i}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega\right)$.

Subsequently, the n system was solved. To derive the ARL, the matrix relation can possibly be represented as follows:

$\hat{\zeta}_{n\times 1} = 1_{n\times 1} + R_{n\times n}\hat{\zeta}_{n\times 1}, I_{n\times n} - R_{n\times n} = 1_{n\times 1}$ or $\hat{\zeta}_{n\times 1} = (I_n - R_{n\times n})^{-1} \cdot 1_{n\times 1}$. Finally, r_i is instead of v, the NIE approximation of ARL is rewritten following Equation 20 as:

$$\hat{\zeta}(v) \approx 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n} \hat{\zeta}(v_j) \cdot g\left(\frac{r_j-(1-\lambda_1)v}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega\right) \tag{20}$$

2.3. Sensitivity Measurements of Control chart

First, let $\zeta(v)$ and $\hat{\zeta}(v)$ stand for the ARL with NIE approach and the ARL with exact solution, which is calculated by Equation 17 and 20, respectively. Then, the percentage accuracy ($\%Acc$), which shows the relative effectiveness of the two suggested ARL approaches, was calculated using Equation 21.

$$\%Acc = 100 - \left(\left|\frac{\zeta(v)-\hat{\zeta}(v)}{\zeta(v)}\right| \times 100\%\right) \tag{21}$$

The computation is then based on the efficiency of the ARL, with various parameter values chosen according to the DEWMA chart. After that, it is compared to the CUSUM and EEWMA charts. In addition to the average run length (ARL), the run length (RL) distribution is often described using additional measures such as the median run length (MRL) and the standard deviation of the run length ($SDRL$), which provide further insights into chart performance. Thus,

$$ARL_0 = \frac{1}{\alpha},\ MRL_0 = \frac{Log(0.5)}{Log(1-\alpha)},\ SDRL_0 = \sqrt{\frac{1-\alpha}{\alpha^2}} \tag{22}$$

where type I error represents $\alpha = 1 - P(e < X_t < f|\gamma_0)$.

In this study, $ARL_0$was fixed at 500. Form the ARL_0value that can be calculated as $MRL_0$and $SDRL_0$by Equation 22 at approximately 346 and 500, respectively. Subsequently, MRL_1 and $SDRL_1$ are calculated using the formulas presented in Equation 23 below.

$$ARL_1 = \frac{1}{1-\beta},\ MRL_1 = \frac{Log(0.5)}{Log(\beta)},\ SDRL_1 = \sqrt{\frac{\beta}{(1-\beta)^2}} \tag{23}$$

where type II error represents $\beta = 1 - P(e < X_t < f|\gamma_1)$.

The Least ARL_1, MRL_1 and $SDRL_1$ values were presented the best performance of control charts [20, 22].

2.4. Existence and Uniqueness of Exact ARL for Demonstration

To demonstrate the existence and uniqueness of the ARL solution, this research employs Banach's fixed-point theorem [23, 24]. Since the explicit ARL formula must satisfy both existence and uniqueness conditions, Banach's theorem provides theoretical support for the solution. For the class of all continuous functions, let T represent the operation, which can be defined as follows:

$$T(\zeta(v)) = 1 + \frac{1}{\lambda_1\lambda_2}\int_e^f \zeta(\rho) \cdot g\left(\frac{\rho-(1-\lambda_1)v}{\lambda_1\lambda_2} - \frac{(1-\lambda_2)z}{\lambda_2} - \omega\right) d\rho \tag{24}$$

Theorem 1: Banach's Fixed-point Theorem: Let (X, d) and $T: X \to X$ are the complete metric space and the contraction mapping, respectively. Moreover, T is referred to unique on fixed point; thus, there exists a unique solution to the fixed point when $T(\zeta(v)) = \zeta(v) \in X$. To demonstrate that, let T be the contraction mapping for $\zeta(v)_1, \zeta(v)_2 \in Q[e, f]$ Such that, $\|T(\zeta(v)_1) - T(\zeta(v)_2)\| \le Q\|\zeta(v)_1 - \zeta(v)_2\|$, and $\zeta(v)_1, \zeta(v)_2 \in X$, where Q is a positive constant with $0 \le Q < 1$ under the norm $\|\zeta(v)\|_\infty = sup_{v\in[e,f]}|\zeta(v)|$. By considering:

$$
\begin{aligned}
&\|T(\zeta(v)_1) - T(\zeta(v)_2)\|_\infty = sup_{v\in[e,f]}|\zeta(v)_1 - \zeta(v)_2| \\
&= sup_{v\in[e,f]}\left|\frac{\Lambda(v)}{\gamma\lambda_1\lambda_2}\right| \int_e^f \left((\zeta(v)_1 - \zeta(v)_2)\cdot M_0(\rho)\right) dp \le sup_{v\in[e,f]}\|T(\zeta(v)_1) - T(\zeta(v)_2)\|_\infty \, \Lambda(v)(M_0(f) - M_0(e)) \\
&= \|T(\zeta(v)_1) - T(\zeta(v)_2)\|_\infty \, sup_{v\in[e,f]}|\Lambda(v)|\,|M_0(f) - M_0(e)| \le Q\|T(\zeta(v)_1) - T(\zeta(v)_2)\|_\infty
\end{aligned}
\tag{25}
$$

wheren $Q = sup_{v\in[e,f]}|\Lambda(v)|\,|M_0(f) - M_0(e)|; Q \in [0,1]$.

Moreover, using the NIE technique, the number of division points needed to estimate the ARL at $n = 500$ is found. ARL_0, the process in-control, was computed.

3. The ARL Procedure for Analyzing Outcomes

3.1. The Exact Solution of ARL for the Quadratic Trend AR(P) Model Running on the Control Charts

Input:

- Set parameters of quadratic trend AR(p): $\psi, \eta, \vartheta, \phi_i$ in $X_t = \psi + \eta t + \vartheta t^2 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \xi_t$
- Set parameters for control charts: λ_1 =0.05, 0.10, 0.15, λ_2 =0.6λ_1, λ_1, 1.6$\lambda_1$for DEWMA chart, λ_2 =0.2λ_1, 0.6λ_1 for EEWMA chart, and $\kappa > 0$ for CUSUM chart
- Set $\gamma = \gamma_0$ for in-control process, then set $\gamma_0 = 1$ when using simulated data, and set $\gamma_0$equal to the exponential white noise ($\xi_t \sim Exp(\gamma)$) when using a real-world dataset, and define $ARL_0 = 500$.
- Set $\gamma = \gamma_1 = \gamma_0(1+\delta)$for out-of-control process and set $\delta = 0.001{,}0.002{,}0.003{,}0.005{,}0.01{,}0.03{,}0.1{,}0.5$

Output:

- Obtain the upper control limit (UCL) of the control chart under various scenarios of the specified parameters at $ARL_0 = 500$
- Obtain the ARL_1which derived from the out-of-control process, by determining δ as specified above.

Furthermore, the solution can be derived using the approach illustrated in Figure 1, as outlined next.

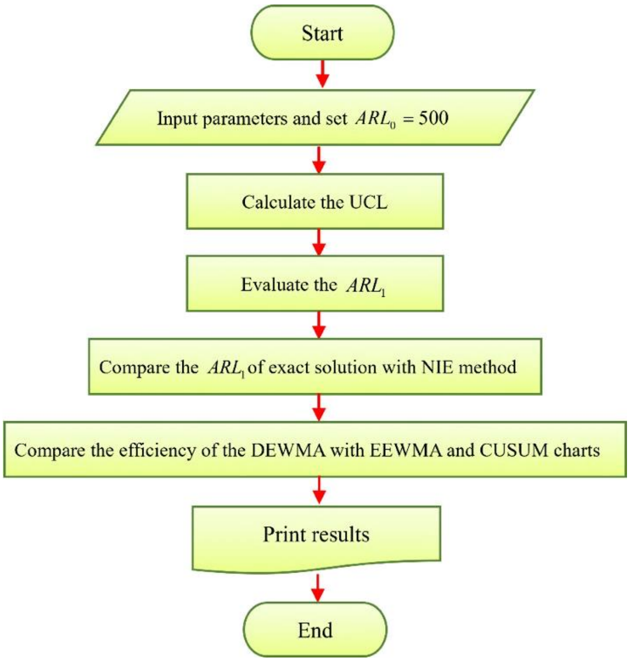


Figure 1. The Process of Methodology of evaluating *ARL*

4. The Outcomes of the Performance Evaluation

To estimate the ARL at $n = 500$ the number of division points required is determined using the NIE technique. The results demonstrated the ARL's ability to detect shifts in the process mean using the DEWMA chart, as presented in Table 1 for quadratic trend AR(2) model and Table 2. for quadratic trend AR(3) model. All scenarios exhibit extraordinarily high the percentage accuracy ($\%Acc$), nearly 100%, according to the ARL_1 results. This indicates that there is no difference between the two approaches in terms of accuracy. However, the exact solution appears fairly quickly in every scenario, while the ARL_1generated using the NIE approach takes roughly 2 to 3 seconds to compute for the two-sided DEWMA chart at $LCL = e = 0.001$. This indicates that there is only a slight difference between the two approaches in terms of computation time. Moreover, the results obtained from both methods were computed using a system running Windows 10 (64-bit) with an Intel Core i5-8250U processor (1.60 GHz, up to 1.80 GHz) and 4 GB of RAM. Moreover, while the exact solution does not depend on the specifications of the CPU, the computation time of the NIE technique is influenced by the CPU's performance. Therefore, it is reasonable to proceed with these exact formulas.

Table 1. ARL_1 values of the exact formula and NIE technique for Quadratic trend AR(2) model on DEWMA control chart with known parameters; $\lambda_1 = 0.05, \psi = \eta = 0.1, \vartheta = -0.8$ at $ARL_0 = 500$, and [e, f] = [0.001, f]

ϕ_2	Shift size δ	ϕ_1	0.1			-0.1		
		λ_2	$0.6\lambda_1$	λ_1	$1.6\lambda_1$	$0.6\lambda_1$	λ_1	$1.6\lambda_1$
		f	**0.001082563**	**0.001516962**	**0.002747883**	**0.001100874**	**0.001632161**	**0.003140205**
0.2	0.001	$\zeta(v)$	105.723 (<0.01)	163.560 (<0.01)	207.867 (<0.01)	109.668 (<0.01)	170.223 (<0.01)	217.026 (<0.01)
		$\hat{\zeta}(v)$	105.723 (2.766)	163.560 (2.875)	207.867 (2.812)	109.668 (2.703)	170.223 (2.876)	217.026 (2.828)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.002	$\zeta(v)$	59.469 (<0.01)	98.066 (<0.01)	131.499 (<0.01)	61.949 (<0.01)	102.879 (<0.01)	138.876 (<0.01)
		$\hat{\zeta}(v)$	59.469 (2.813)	98.066 (2.813)	131.499 (2.828)	61.949 (2.811)	102.879 (2.828)	138.876 (2.781)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.003	$\zeta(v)$	41.546 (<0.01)	70.190 (<0.01)	96.326 (<0.01)	43.343 (<0.01)	73.881 (<0.01)	102.267 (<0.01)
		$\hat{\zeta}(v)$	41.546 (2.813)	70.190 (2.876)	96.326 (2.796)	43.343 (2.765)	73.881 (2.844)	102.267 (2.797)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.005	$\zeta(v)$	26.123 (<0.01)	44.947 (<0.01)	62.953 (<0.01)	27.281 (<0.01)	47.447 (<0.01)	67.166 (<0.01)
		$\hat{\zeta}(v)$	26.123 (2.750)	44.947 (2.828)	62.953 (2.844)	27.281 (2.843)	47.447 (2.843)	67.166 (2.750)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.01	$\zeta(v)$	13.859 (<0.01)	23.983 (<0.01)	34.063 (<0.01)	14.475 (<0.01)	25.367 (<0.01)	36.488 (<0.01)
		$\hat{\zeta}(v)$	13.859 (2.813)	23.983 (2.797)	34.063 (2.844)	14.475 (2.844)	25.367 (2.797)	36.488 (2.782)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.03	$\zeta(v)$	5.325 (<0.01)	8.901 (<0.01)	12.583 (<0.01)	5.545 (<0.01)	9.410 (<0.01)	13.499 (<0.01)
		$\hat{\zeta}(v)$	5.325 (2.875)	8.901 (2.796)	12.583 (2.890)	5.545 (2.828)	9.410 (2.922)	13.499 (2.797)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.1	$\zeta(v)$	2.268 (<0.01)	3.394 (<0.01)	4.584 (<0.01)	2.341 (<0.01)	3.567 (<0.01)	4.899 (<0.01)
		$\hat{\zeta}(v)$	2.268 (2.828)	3.394 (2.859)	4.584 (2.813)	2.341 (2.765)	3.567 (2.797)	4.899 (2.828)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.5	$\zeta(v)$	1.219 (<0.01)	1.473 (<0.01)	1.762 (<0.01)	1.239 (<0.01)	1.522 (<0.01)	1.853 (<0.01)
		$\hat{\zeta}(v)$	1.219 (2.858)	1.473 (2.811)	1.762 (2.844)	1.239 (2.781)	1.522 (2.874)	1.853 (2.812)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
		f	**0.0011232545**	**0.001773234**	**0.00362206**	**0.0011506133**	**0.001946093**	**0.00421463**
-0.2	0.001	$\zeta(v)$	113.856 (<0.01)	177.359 (<0.01)	227.184 (<0.01)	118.337 (<0.01)	185.015 (<0.01)	238.871 (<0.01)
		$\hat{\zeta}(v)$	113.856 (2.796)	177.359 (2.829)	227.184 (2.828)	118.337 (2.797)	185.015 (2.796)	238.871 (2.812)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.002	$\zeta(v)$	64.609 (<0.01)	108.113 (<0.01)	147.266 (<0.01)	67.477 (<0.01)	113.834 (<0.01)	157.160 (<0.01)
		$\hat{\zeta}(v)$	64.609 (2.813)	108.113 (2.797)	147.266 (2.766)	67.477 (2.797)	113.834 (2.812)	157.160 (2.812)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.003	$\zeta(v)$	45.278 (<0.01)	77.922 (<0.01)	109.104 (<0.01)	47.372 (<0.01)	82.376 (<0.01)	117.265 (<0.01)
		$\hat{\zeta}(v)$	45.278 (2.922)	77.922 (2.812)	109.104 (2.781)	47.372 (2.796)	82.376 (2.828)	117.265 (2.859)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.005	$\zeta(v)$	28.533 (<0.01)	50.202 (<0.01)	72.069 (<0.01)	29.890 (<0.01)	53.261 (<0.01)	77.991 (<0.01)
		$\hat{\zeta}(v)$	28.533 (2.797)	50.202 (2.844)	72.069 (2.844)	29.890 (2.828)	53.261 (2.797)	77.991 (2.782)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.01	$\zeta(v)$	15.142 (<0.01)	26.901 (<0.01)	39.336 (<0.01)	15.867 (<0.01)	28.614 (<0.01)	42.813 (<0.01)
		$\hat{\zeta}(v)$	15.142 (2.813)	26.901 (2.765)	39.336 (2.719)	15.867 (2.828)	28.614 (2.797)	42.813 (2.812)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.03	$\zeta(v)$	5.783 (<0.01)	9.977 (<0.01)	14.582 (<0.01)	6.043 (<0.01)	10.613 (<0.01)	15.913 (<0.01)
		$\hat{\zeta}(v)$	5.783 (2.984)	9.977 (2.781)	14.582 (2.781)	6.043 (2.844)	10.613 (2.844)	15.913 (2.781)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.1	$\zeta(v)$	2.422 (<0.01)	3.760 (<0.01)	5.270 (<0.01)	2.509 (<0.01)	3.978 (<0.01)	5.726 (<0.01)
		$\hat{\zeta}(v)$	2.422 (2.813)	3.760 (2.797)	5.270 (2.858)	2.509 (2.812)	3.978 (2.797)	5.726 (2.781)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.5	$\zeta(v)$	1.260 (<0.01)	1.578 (<0.01)	1.960 (<0.01)	1.285 (<0.01)	1.642 (<0.01)	2.090 (<0.01)
		$\hat{\zeta}(v)$	1.260 (2.797)	1.578 (2.781)	1.960 (2.813)	1.285 (2.797)	1.642 (2.781)	2.090 (2.750)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00

Table 2. ARL_1 values of the exact formula and NIE technique for Quadratic trend AR(3) model on DEWMA control chart with known parameters; $\lambda_1 = 0.05, \psi = \eta = 0.1, \vartheta = -0.8$ at $ARL_0 = 500$, and [e, f] = [0.001, f]

ϕ_3	Shift size δ	ϕ_2	0.2			-0.2		
		λ_2	$0.6\lambda_1$	λ_1	$1.6\lambda_1$	$0.6\lambda_1$	λ_1	$1.6\lambda_1$
		f	**0.0010611419**	**0.001382449**	**0.0022911**	**0.0010912596**	**0.00157165**	**0.002934**
0.3	0.001	$\zeta(v)$	100.225 (<0.01)	154.264 (<0.01)	195.631 (<0.01)	107.644 (<0.01)	166.827 (<0.01)	212.350 (<0.01)
		$\hat{\zeta}(v)$	100.225 (2.967)	154.264 (2.985)	195.631 (3.016)	107.644 (2.859)	166.827 (2.891)	212.350 (2.969)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.002	$\zeta(v)$	56.056 (<0.01)	91.497 (<0.01)	121.849 (<0.01)	60.680 (<0.01)	100.420 (<0.01)	135.085 (<0.01)
		$\hat{\zeta}(v)$	56.056 (2.937)	91.497 (2.938)	121.849 (2.875)	60.680 (2.953)	100.420 (2.937)	135.085 (2.844)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.003	$\zeta(v)$	39.084 (<0.01)	65.197 (<0.01)	88.634 (<0.01)	42.424 (<0.01)	71.993 (<0.01)	99.204 (<0.01)
		$\hat{\zeta}(v)$	39.084 (2.922)	65.197 (2.937)	88.634 (2.874)	42.424 (2.937)	71.993 (2.844)	99.204 (2.906)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.005	$\zeta(v)$	24.544 (<0.01)	41.592 (<0.01)	57.554 (<0.01)	26.690 (<0.01)	46.167 (<0.01)	64.988 (<0.01)
		$\hat{\zeta}(v)$	24.544 (2.938)	41.592 (2.906)	57.554 (2.984)	26.690 (2.921)	46.167 (2.890)	64.988 (2.890)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.01	$\zeta(v)$	13.023 (<0.01)	22.139 (<0.01)	30.984 (<0.01)	14.161 (<0.01)	24.658 (<0.01)	35.232 (<0.01)
		$\hat{\zeta}(v)$	13.023 (3.000)	22.139 (2.907)	30.984 (2.907)	14.161 (2.797)	24.658 (2.843)	35.232 (2.937)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.03	$\zeta(v)$	5.028 (<0.01)	8.228 (<0.01)	11.427 (<0.01)	5.433 (<0.01)	9.149 (<0.01)	13.023 (<0.01)
		$\hat{\zeta}(v)$	5.028 (2.938)	8.228 (2.922)	11.427 (2.922)	5.433 (2.890)	9.149 (2.750)	13.023 (2.860)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.1	$\zeta(v)$	2.169 (<0.01)	3.166 (<0.01)	4.189 (<0.01)	2.304 (<0.01)	3.478 (<0.01)	4.735 (<0.01)
		$\hat{\zeta}(v)$	2.169 (2.953)	3.166 (2.797)	4.189 (2.875)	2.304 (2.890)	3.478 (2.921)	4.735 (2.921)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.5	$\zeta(v)$	1.193 (<0.01)	1.409 (<0.01)	1.650 (<0.01)	1.228 (<0.01)	1.497 (<0.01)	1.806 (<0.01)
		$\hat{\zeta}(v)$	1.193 (2.968)	1.409 (2.907)	1.650 (2.937)	1.228 (2.890)	1.497 (3.469)	1.806 (2.891)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
		f	**0.001111503**	**0.001699123**	**0.00336873**	**0.0011664984**	**0.002046665**	**0.00456048**
-0.3	0.001	$\zeta(v)$	111.711 (<0.01)	173.730 (<0.01)	222.020 (<0.01)	120.703 (<0.01)	189.133 (<0.01)	245.298 (<0.01)
		$\hat{\zeta}(v)$	111.711 (2.891)	173.730 (2.937)	222.020 (2.874)	120.703 (2.969)	189.133 (2.937)	245.298 (2.875)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.002	$\zeta(v)$	63.249 (<0.01)	105.440 (<0.01)	142.948 (<0.01)	68.998 (<0.01)	116.934 (<0.01)	162.813 (<0.01)
		$\hat{\zeta}(v)$	63.249 (2.844)	105.440 (2.937)	142.948 (2.844)	68.998 (2.813)	116.934 (2.828)	162.813 (2.890)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.003	$\zeta(v)$	44.289 (<0.01)	75.854 (<0.01)	105.567 (<0.01)	48.484 (<0.01)	84.798 (<0.01)	122.004 (<0.01)
		$\hat{\zeta}(v)$	44.289 (2.921)	75.854 (2.844)	105.567 (2.875)	48.484 (2.876)	84.798 (2.859)	122.004 (2.921)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.005	$\zeta(v)$	27.894 (<0.01)	48.790 (<0.01)	69.521 (<0.01)	30.613 (<0.01)	54.932 (<0.01)	81.481 (<0.01)
		$\hat{\zeta}(v)$	27.894 (2.938)	48.790 (2.999)	69.521 (2.953)	30.613 (2.922)	54.932 (2.922)	81.481 (2.859)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.01	$\zeta(v)$	14.801 (<0.01)	26.114 (<0.01)	37.851 (<0.01)	16.253 (<0.01)	29.553 (<0.01)	44.887 (<0.01)
		$\hat{\zeta}(v)$	14.801 (2.906)	26.114 (2.923)	37.851 (2.907)	16.253 (2.797)	29.553 (2.796)	44.887 (2.891)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.03	$\zeta(v)$	5.661 (<0.01)	9.686 (<0.01)	14.016 (<0.01)	6.182 (<0.01)	10.963 (<0.01)	16.712 (<0.01)
		$\hat{\zeta}(v)$	5.661 (2.814)	9.686 (2.907)	14.016 (2.922)	6.182 (2.859)	10.963 (2.906)	16.712 (2.907)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.1	$\zeta(v)$	2.381 (<0.01)	3.660 (<0.01)	5.076 (<0.01)	2.556 (<0.01)	4.098 (<0.01)	5.998 (<0.01)
		$\hat{\zeta}(v)$	2.381 (2.906)	3.660 (2.922)	5.076 (2.859)	2.556 (2.892)	4.098 (2.922)	5.998 (2.813)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00
	0.5	$\zeta(v)$	1.249 (<0.01)	1.549 (<0.01)	1.904 (<0.01)	1.298 (<0.01)	1.678 (<0.01)	2.166 (<0.01)
		$\hat{\zeta}(v)$	1.249 (2.984)	1.549 (2.907)	1.904 (2.890)	1.298 (2.922)	1.678 (2.937)	2.166 (2.890)
		$\%Acc$	100.00	100.00	100.00	100.00	100.00	100.00

4.1. Performance Evaluation of Simulated Data for the Control Chart

This section examines the explicit ARL of the DEWMA chart and compares it to EEWMA and CUSUM charts under AR(2) and AR(3) processes with quadratic trends. For the DEWMA chart, the smoothing parameter $\lambda_1$was set at 0.05, 0.10, and 0.15, while λ_2 took on values determined$4\lambda_1, 1.6\lambda_1, \lambda_1$,and $0.6\lambda_1$, which denoted as DEWMA-1, DEWMA-2, DEWMA-3, and DEWMA-4, respectively. The EEWMA chart, on the other hand, was evaluated using specified $\lambda_2$values of $0.2\lambda_1$and$0.6\lambda_1$, which are represented as EEWMA-1, and EEWMA-2, respectively. For the in-control scenario, the ARL_0 value was fixed at 500. The effectiveness of the control charts was assessed using ARL_1, $SDRL_1$, and MRL_1 values. Tables 3 and 4 present the comparative results of the CUSUM, EEWMA, and DEWMA charts under various scenarios, specifically for the quadratic trend AR(2) and AR(3) models, respectively. The outcomes indicate that a lower value of λ_1 leads to a decrease in ARL_1, $SDRL_1$, and MRL_1 values. The DEWMA chart consistently outperforms the EEWMA and CUSUM charts in detecting small shift changes with $0 < \delta \leq 0.5$. Moreover, lower value of λ_2, which is close to λ_1, as evidenced by$\lambda_2 = 0.6\lambda_1$and for all λ_1 considered in this research, this leads to a decrease inARL_1, $SDRL_1$, and MRL_1 values for both the EEWMA and DEWMA charts, under both the quadratic trend AR(2) and AR(3) models. The results of this study suggest that selecting λ_2 values closer to λ_1 can significantly enhance the effectiveness in detecting small process shifts in both EEWMA and DEWMA charts. In addition, lower values of λ_1 also demonstrate efficiency in reducing the run length (RL) evaluations. Therefore, under the conditions and procedures adopted in this study, using a lower exponential smoothing value is recommended to enhance the detection capability and overall process monitoring performance. It is noted that the lowest ARL_1, $SDRL_1$, and MRL_1 values for all λ_1 conditions in both scenarios of the quadratic trend AR(2) and AR(3) models are shown in bold and italic in Tables 3 to 5.

Table 3. RL_1 values of exact formula running on two-sided control charts for the quadratic trend AR(2) model with known parameters; $\psi = \eta = 0.2, \vartheta = -0.1, \phi_1 = 0.1, \phi_2 = 0.2$, and [LCL, UCL] = [0.001, UCL]

λ_1	Control chart		UCL	δ	0.001	0.002	0.003	0.005	0.01	0.03	0.1	0.5
0.05	CUSUM	$\kappa = 3$	4.154	ARL_1	496.69	493.23	489.81	483.05	466.67	407.89	264.70	49.77
				$SDRL_1$	496.19	492.73	489.31	482.55	466.17	407.39	264.20	49.27
				MRL_1	343.93	341.54	339.16	334.48	323.13	282.38	183.13	34.15
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.03495131	ARL_1	255.75	172.06	129.78	87.20	48.22	17.88	6.27	2.13
				$SDRL_1$	255.25	171.56	129.28	86.69	47.72	17.37	5.75	1.55
				MRL_1	176.93	118.92	89.61	60.09	33.08	12.04	3.99	1.09
		$\lambda_2 = 0.6\lambda_1$	0.05146347	ARL_1	279.49	194.18	148.91	101.75	57.14	21.39	7.47	2.46
				$SDRL_1$	278.99	193.68	148.41	101.25	56.64	20.89	6.95	1.89
				MRL_1	193.38	134.25	102.87	70.18	39.26	14.48	4.82	1.33
	DEWMA	$\lambda_2 = 4\lambda_1$	0.00472606	ARL_1	218.45	140.01	103.16	67.77	36.78	13.53	4.83	1.77
				$SDRL_1$	217.95	139.51	102.66	67.27	36.28	13.02	4.30	1.17
				MRL_1	151.07	96.70	71.16	46.63	25.15	9.03	2.99	0.83
		$\lambda_2 = 1.6\lambda_1$	0.001705922	ARL_1	174.22	105.77	76.09	48.92	26.14	9.63	3.58	1.48
				$SDRL_1$	173.72	105.27	75.59	48.42	25.63	9.12	3.04	0.84
				MRL_1	120.41	72.97	52.39	33.56	17.77	6.32	2.12	0.62
		$\lambda_2 = \lambda_1$	0.0012095211	ARL_1	137.94	80.32	56.82	36.03	19.11	7.13	2.80	1.31
				$SDRL_1$	137.44	79.82	56.31	35.52	18.60	6.61	2.24	0.64
				MRL_1	95.26	55.33	39.03	24.62	12.90	4.59	1.57	0.48
		$\lambda_2 = 0.6\lambda_1$	0.00103353962	ARL_1	***90.62***	***50.19***	***34.88***	***21.86***	***11.61***	***4.53***	***2.00***	***1.15***
				$SDRL_1$	***90.12***	***49.69***	***34.38***	***21.36***	***11.10***	***4.00***	***1.42***	***0.42***
				MRL_1	***62.47***	***34.44***	***23.83***	***14.80***	***7.70***	***2.78***	***1.00***	***0.34***
0.10	CUSUM	$\kappa = 3$	4.154	ARL_1	496.69	493.23	489.81	483.05	466.67	407.89	264.70	49.77
				$SDRL_1$	496.19	492.73	489.31	482.55	466.17	407.39	264.20	49.27
				MRL_1	343.93	341.54	339.16	334.48	323.13	282.38	183.13	34.15
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.06984855	ARL_1	259.44	175.40	132.62	89.32	49.50	18.36	6.42	2.16
				$SDRL_1$	258.94	174.90	132.12	88.82	49.00	17.86	5.90	1.58
				MRL_1	179.48	121.23	91.58	61.57	33.96	12.38	4.09	1.11
		$\lambda_2 = 0.6\lambda_1$	0.1029645	ARL_1	282.71	197.30	151.66	103.88	58.45	21.90	7.63	2.49
				$SDRL_1$	282.21	196.80	151.15	103.37	57.95	21.40	7.11	1.92
				MRL_1	195.61	136.41	104.77	71.65	40.17	14.83	4.93	1.35

	DEWMA	$\lambda_2 = 4\lambda_1$	0.02037356	ARL_1	240.39	158.46	118.32	78.70	43.14	15.91	5.59	1.94
				$SDRL_1$	239.88	157.96	117.81	78.20	42.64	15.40	5.06	1.35
				MRL_1	166.28	109.49	81.66	54.21	29.56	10.67	3.51	0.96
		$\lambda_2 = 1.6\lambda_1$	0.00630545	ARL_1	219.43	140.81	103.80	68.21	37.01	13.59	4.82	1.75
				$SDRL_1$	218.93	140.31	103.30	67.70	36.51	13.08	4.29	1.14
				MRL_1	151.75	97.25	71.60	46.93	25.31	9.07	2.98	0.82
		$\lambda_2 = \lambda_1$	0.003275698	ARL_1	200.71	125.81	91.75	59.70	32.15	11.78	4.23	1.60
				$SDRL_1$	200.21	125.31	91.25	59.20	31.64	11.27	3.70	0.98
				MRL_1	138.78	86.86	63.25	41.03	21.93	7.81	2.57	0.71
		$\lambda_2 = 0.6\lambda_1$	0.0012043922	ARL_1	***141.73***	***82.86***	***58.68***	***37.23***	***19.72***	***7.29***	***2.80***	***1.28***
				$SDRL_1$	***141.23***	***82.36***	***58.18***	***36.73***	***19.21***	***6.78***	***2.24***	***0.60***
				MRL_1	***97.89***	***57.08***	***40.33***	***25.46***	***13.32***	***4.70***	***1.57***	***0.45***
0.15	CUSUM	$\kappa = 3$	4.154	ARL_1	496.69	493.23	489.81	483.05	466.67	407.89	264.70	49.77
				$SDRL_1$	496.19	492.73	489.31	482.55	466.17	407.39	264.20	49.27
				MRL_1	343.93	341.54	339.16	334.48	323.13	282.38	183.13	34.15
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.105747	ARL_1	262.36	178.07	134.90	91.03	50.53	18.75	6.54	2.18
				$SDRL_1$	261.85	177.56	134.40	90.53	50.03	18.25	6.02	1.61
				MRL_1	181.50	123.08	93.16	62.75	34.68	12.65	4.18	1.13
		$\lambda_2 = 0.6\lambda_1$	0.155548	ARL_1	285.15	285.15	285.15	285.15	285.15	285.15	285.15	285.15
				$SDRL_1$	284.65	199.17	153.25	105.00	58.96	21.79	7.23	1.95
				MRL_1	197.30	138.05	106.23	72.78	40.87	15.11	5.02	1.37
	DEWMA	$\lambda_2 = 4\lambda_1$	0.0490872	ARL_1	248.43	165.49	124.20	83.03	45.71	16.88	5.90	2.01
				$SDRL_1$	247.93	164.99	123.70	82.53	45.20	16.37	5.38	1.42
				MRL_1	171.85	114.36	85.74	57.21	31.33	11.35	3.73	1.01
		$\lambda_2 = 1.6\lambda_1$	0.0158829	ARL_1	234.19	153.14	113.90	75.48	41.24	15.17	5.33	1.86
				$SDRL_1$	233.69	152.64	113.40	74.98	40.73	14.66	4.80	1.27
				MRL_1	161.98	105.80	78.60	51.97	28.24	10.16	3.34	0.90
		$\lambda_2 = \lambda_1$	0.00821218	ARL_1	221.34	142.35	105.04	69.08	37.51	13.76	4.86	1.75
				$SDRL_1$	220.84	141.84	104.53	68.58	37.00	13.25	4.33	1.14
				MRL_1	153.08	98.32	72.46	47.54	25.65	9.19	3.01	0.82
		$\lambda_2 = 0.6\lambda_1$	0.002055776	ARL_1	***178.35***	***108.79***	***78.39***	***50.45***	***26.94***	***9.86***	***3.59***	***1.44***
				$SDRL_1$	***177.85***	***108.28***	***77.89***	***49.95***	***26.44***	***9.34***	***3.05***	***0.80***
				MRL_1	***123.28***	***75.06***	***53.99***	***34.62***	***18.33***	***6.48***	***2.12***	***0.58***

Table 4. RL_1 values of exact formula running on two-sided control charts for the quadratic trend AR(3) model with known parameters; $\psi = \eta = 0.2, \vartheta = -0.1, \phi_1 = \phi_2 = 0.1$, and $\phi_3 = -0.2$, and [LCL, UCL] = [0.001, UCL]

λ_1	Control chart		UCL	δ	0.001	0.002	0.003	0.005	0.01	0.03	0.1	0.5
0.05	CUSUM	$\kappa = 3$	3.719	ARL_1	496.95	493.63	490.33	483.83	468.05	411.22	271.23	53.54
				$SDRL_1$	496.45	493.13	489.83	483.33	467.55	410.72	270.73	53.04
				MRL_1	344.11	341.81	339.53	335.02	324.08	284.69	187.65	36.76
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.03495131	ARL_1	274.24	189.16	144.51	98.35	55.03	20.55	7.18	2.38
				$SDRL_1$	273.74	188.66	144.00	97.85	54.53	20.04	6.66	1.81
				MRL_1	189.74	130.77	99.82	67.83	37.80	13.90	4.62	1.27
		$\lambda_2 = 0.6\lambda_1$	0.06937324	ARL_1	306.83	221.53	173.47	121.17	69.44	26.33	9.11	2.85
				$SDRL_1$	306.33	221.03	172.97	120.66	68.94	25.82	8.59	2.30
				MRL_1	212.33	153.21	119.89	83.64	47.79	17.90	5.96	1.61
	DEWMA	$\lambda_2 = 4\lambda_1$	0.006046663	ARL_1	231.80	151.13	112.26	74.31	40.59	14.98	5.32	1.91
				$SDRL_1$	231.30	150.63	111.76	73.81	40.09	14.47	4.80	1.32
				MRL_1	160.33	104.41	77.46	51.16	27.79	10.03	3.33	0.93

		$\lambda_2 = 1.6\lambda_1$	0.001954412	ARL_1	184.36	113.31	81.94	52.93	28.38	10.46	3.86	1.56
				$SDRL_1$	183.86	112.81	81.44	52.42	27.87	9.95	3.32	0.93
				MRL_1	127.44	78.19	56.45	36.34	19.32	6.90	2.31	0.67
		$\lambda_2 = \lambda_1$	0.001283037	ARL_1	145.73	85.61	60.76	38.64	20.53	7.64	2.97	1.36
				$SDRL_1$	145.23	85.11	60.26	38.14	20.02	7.12	2.42	0.69
				MRL_1	100.67	58.99	41.77	26.43	13.88	4.94	1.69	0.52
		$\lambda_2 = 0.6\lambda_1$	0.0010452828	ARL_1	***95.20***	***52.98***	***36.87***	***23.13***	***12.28***	***4.76***	***2.08***	***1.17***
				$SDRL_1$	***94.70***	***52.47***	***36.37***	***22.63***	***11.77***	***4.23***	***1.50***	***0.45***
				MRL_1	***65.64***	***36.37***	***25.21***	***15.68***	***8.16***	***2.94***	***1.06***	***0.36***
0.10	CUSUM	$\kappa = 3$	3.719	ARL_1	496.95	493.63	490.33	483.83	468.05	411.22	271.23	53.54
				$SDRL_1$	496.45	493.13	489.83	483.33	467.55	410.72	270.73	53.04
				MRL_1	344.11	341.81	339.53	335.02	324.08	284.69	187.65	36.76
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.09487355	ARL_1	279.02	193.73	148.51	101.43	56.92	21.28	7.41	2.42
				$SDRL_1$	278.52	193.23	148.01	100.93	56.42	20.78	6.89	1.85
				MRL_1	193.05	133.94	102.59	69.96	39.11	14.40	4.78	1.30
		$\lambda_2 = 0.6\lambda_1$	0.13966656	ARL_1	311.16	226.07	177.64	124.55	71.63	27.21	9.39	2.91
				$SDRL_1$	310.66	225.56	177.14	124.05	71.13	26.70	8.87	2.36
				MRL_1	215.33	156.35	122.79	85.98	49.31	18.51	6.15	1.64
	DEWMA	$\lambda_2 = 4\lambda_1$	0.02738234	ARL_1	256.21	172.47	130.12	87.45	48.37	17.93	6.27	2.12
				$SDRL_1$	255.71	171.97	129.62	86.95	47.87	17.42	5.75	1.54
				MRL_1	177.25	119.20	89.85	60.27	33.18	12.08	3.99	1.09
		$\lambda_2 = 1.6\lambda_1$	0.00820453	ARL_1	232.78	151.94	112.92	74.78	40.84	15.04	5.31	1.88
				$SDRL_1$	232.28	151.44	112.42	74.27	40.34	14.54	4.79	1.29
				MRL_1	161.00	104.97	77.92	51.48	27.96	10.08	3.33	0.91
		$\lambda_2 = \lambda_1$	0.00408444	ARL_1	212.45	135.10	99.18	64.93	35.13	12.90	4.61	1.70
				$SDRL_1$	211.95	134.60	98.68	64.42	34.63	12.39	4.08	1.09
				MRL_1	146.91	93.30	68.40	44.66	24.00	8.59	2.83	0.78
		$\lambda_2 = 0.6\lambda_1$	0.001276152	ARL_1	***149.07***	***87.86***	***62.43***	***39.71***	***21.07***	***7.78***	***2.96***	***1.32***
				$SDRL_1$	***148.57***	***87.36***	***61.93***	***39.21***	***20.56***	***7.26***	***2.41***	***0.65***
				MRL_1	***102.98***	***60.55***	***42.93***	***27.18***	***14.25***	***5.04***	***1.68***	***0.49***
0.15	CUSUM	$\kappa = 3$	3.719	ARL_1	496.95	493.63	490.33	483.83	468.05	411.22	271.23	53.54
				$SDRL_1$	496.45	493.13	489.83	483.33	467.55	410.72	270.73	53.04
				MRL_1	344.11	341.81	339.53	335.02	324.08	284.69	187.65	36.76
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.1445905	ARL_1	283.12	197.69	151.99	104.13	58.59	21.93	7.61	2.46
				$SDRL_1$	282.62	197.19	151.49	103.63	58.09	21.43	7.10	1.90
				MRL_1	195.89	136.68	105.01	71.83	40.27	14.85	4.92	1.33
		$\lambda_2 = 0.6\lambda_1$	0.2119926	ARL_1	314.82	229.92	181.21	127.46	73.53	27.98	9.63	2.96
				$SDRL_1$	314.32	229.42	180.71	126.96	73.03	27.47	9.12	2.40
				MRL_1	217.87	159.02	125.26	88.00	50.62	19.04	6.33	1.68
	DEWMA	$\lambda_2 = 4\lambda_1$	0.0668722	ARL_1	266.09	181.48	137.83	93.24	51.87	19.27	6.71	2.22
				$SDRL_1$	265.59	180.98	137.33	92.74	51.37	18.77	6.19	1.65
				MRL_1	184.10	125.45	95.19	64.28	35.61	13.01	4.30	1.16
		$\lambda_2 = 1.6\lambda_1$	0.02131766	ARL_1	249.51	166.46	125.02	83.64	46.08	17.03	5.96	2.03
				$SDRL_1$	249.01	165.96	124.52	83.14	45.58	16.52	5.44	1.45
				MRL_1	172.60	115.03	86.31	57.63	31.59	11.45	3.77	1.02
		$\lambda_2 = \lambda_1$	0.01082036	ARL_1	235.00	153.81	114.46	75.88	41.48	15.27	5.37	1.88
				$SDRL_1$	234.50	153.31	113.95	75.38	40.98	14.76	4.85	1.29
				MRL_1	162.54	106.27	78.99	52.25	28.40	10.23	3.37	0.92
		$\lambda_2 = 0.6\lambda_1$	0.002429622	ARL_1	***188.12***	***116.11***	***84.10***	***54.38***	***29.14***	***10.66***	***3.86***	***1.51***
				$SDRL_1$	***187.62***	***115.61***	***83.60***	***53.87***	***28.63***	***10.15***	***3.32***	***0.87***
				MRL_1	***130.05***	***80.13***	***57.95***	***37.34***	***19.85***	***7.04***	***2.31***	***0.64***

Table 5.RL₁ values of exact formula on two-sided control charts for quadratic trend AR(2) model using the natural gas imports dataset with parameters; $\psi = 0, \eta = 0.332, \vartheta = -0.002, \phi_1 = 0.454, \phi_2 = 0.409$, and [$LCL$, UCL] = [0.001, UCL]

λ_1	Control chart		UCL	δ	0.001	0.002	0.003	0.005	0.01	0.03	0.1	0.5
0.05	CUSUM	$\kappa = 5$	8.265	ARL_1	496.37	492.70	489.07	481.90	464.56	402.66	254.57	44.53
				$SDRL_1$	495.87	492.20	488.57	481.40	464.06	402.16	254.07	44.02
				MRL_1	343.71	341.17	338.65	333.68	321.66	278.75	176.11	30.52
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.05253247	ARL_1	248.52	165.59	124.29	83.11	45.77	16.92	5.93	2.04
				$SDRL_1$	248.02	165.09	123.79	82.61	45.27	16.41	5.41	1.45
				MRL_1	171.91	114.43	85.81	57.26	31.38	11.38	3.76	1.03
		$\lambda_2 = 0.6\lambda_1$	0.06522033	ARL_1	259.21	175.20	132.45	89.20	49.43	18.35	6.43	2.17
				$SDRL_1$	258.71	174.69	131.95	88.70	48.93	17.84	5.91	1.60
				MRL_1	179.33	121.09	91.46	61.48	33.92	12.37	4.10	1.13
	DEWMA	$\lambda_2 = 4\lambda_1$	0.00836681	ARL_1	228.06	147.94	109.62	72.39	39.45	14.52	5.14	1.84
				$SDRL_1$	227.56	147.44	109.12	71.89	38.94	14.01	4.61	1.24
				MRL_1	157.73	102.20	75.64	49.83	26.99	9.71	3.20	0.88
		$\lambda_2 = 1.6\lambda_1$	0.00293538	ARL_1	201.37	126.33	92.18	60.01	32.34	11.88	4.29	1.64
				$SDRL_1$	200.87	125.83	91.67	59.50	31.84	11.37	3.76	1.02
				MRL_1	139.23	87.22	63.54	41.25	22.07	7.88	2.61	0.73
		$\lambda_2 = \lambda_1$	0.001795455	ARL_1	177.36	108.08	77.86	50.12	26.80	9.86	3.65	1.49
				$SDRL_1$	176.86	107.58	77.36	49.62	26.30	9.35	3.11	0.86
				MRL_1	122.59	74.57	53.62	34.40	18.23	6.49	2.16	0.63
		$\lambda_2 = 0.6\lambda_1$	0.00122696	ARL_1	***140.72***	***82.19***	***58.21***	***36.94***	***19.60***	***7.30***	***2.85***	***1.32***
				$SDRL_1$	***140.22***	***81.69***	***57.70***	***36.44***	***19.09***	***6.78***	***2.29***	***0.65***
				MRL_1	***97.19***	***56.62***	***40.00***	***25.26***	***13.23***	***4.70***	***1.60***	***0.49***
0.10	CUSUM	$\kappa = 5$	8.265	ARL_1	496.95	493.63	490.33	483.83	468.05	411.22	271.23	53.54
				$SDRL_1$	496.45	493.13	489.83	483.33	467.55	410.72	270.73	53.04
				MRL_1	344.11	341.81	339.53	335.02	324.08	284.69	187.65	36.76
	EEWMA	$\lambda_2 = 0.2\lambda_1$	0.1052837	ARL_1	251.24	168.01	126.33	84.62	46.66	17.26	6.04	2.05
				$SDRL_1$	250.74	167.50	125.83	84.12	46.16	16.75	5.52	1.47
				MRL_1	173.80	116.11	87.22	58.31	32.00	11.61	3.83	1.04
		$\lambda_2 = 0.6\lambda_1$	0.13037847	ARL_1	261.21	177.02	134.01	90.37	50.14	18.62	6.51	2.19
				$SDRL_1$	260.71	176.52	133.51	89.86	49.63	18.11	5.99	1.61
				MRL_1	180.71	122.35	92.54	62.29	34.40	12.56	4.16	1.14
	DEWMA	$\lambda_2 = 4\lambda_1$	0.03532493	ARL_1	241.36	159.30	119.02	79.22	43.44	16.01	5.62	1.94
				$SDRL_1$	240.86	158.80	118.52	78.71	42.94	15.51	5.09	1.35
				MRL_1	166.95	110.07	82.15	54.56	29.76	10.75	3.54	0.96
		$\lambda_2 = 1.6\lambda_1$	0.01209531	ARL_1	229.32	149.01	110.49	73.01	39.80	14.63	5.16	1.83
				$SDRL_1$	228.82	148.51	109.99	72.51	39.29	14.12	4.63	1.23
				MRL_1	158.61	102.94	76.24	50.26	27.24	9.79	3.22	0.87
		$\lambda_2 = \lambda_1$	0.00660893	ARL_1	218.03	139.65	102.86	67.53	36.62	13.44	4.77	1.73
				$SDRL_1$	217.53	139.15	102.36	67.03	36.12	12.93	4.24	1.12
				MRL_1	150.78	96.45	70.95	46.46	25.04	8.96	2.94	0.80
		$\lambda_2 = 0.6\lambda_1$	0.001965972	ARL_1	***178.87***	***109.18***	***78.70***	***50.68***	***27.08***	***9.92***	***3.63***	***1.46***
				$SDRL_1$	***178.37***	***108.68***	***78.20***	***50.17***	***26.57***	***9.41***	***3.09***	***0.82***
				MRL_1	***123.64***	***75.33***	***54.21***	***34.78***	***18.42***	***6.53***	***2.15***	***0.60***

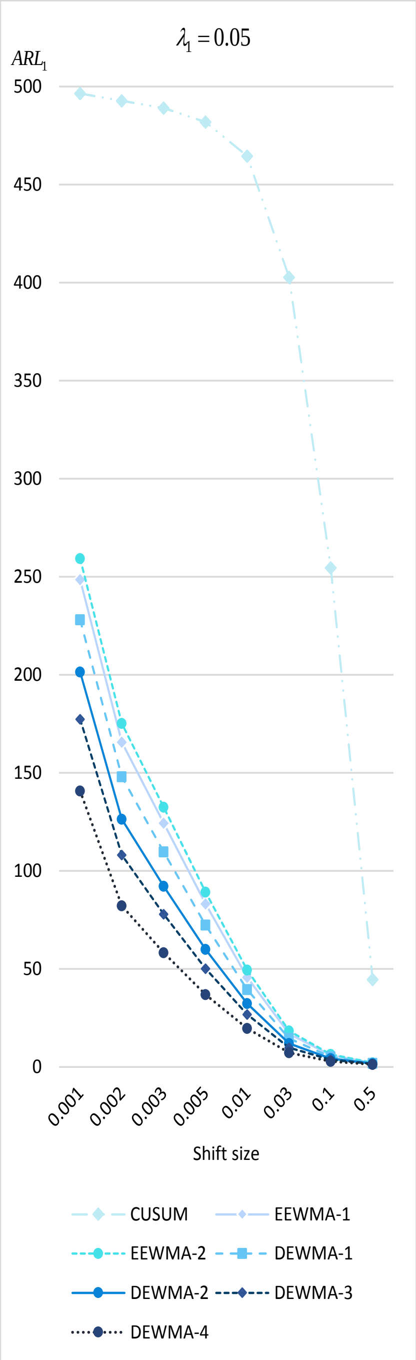
	CUSUM	$\boldsymbol{\kappa = 5}$	8.265	$\boldsymbol{ARL_1}$	496.95	493.63	490.33	483.83	468.05	411.22	271.23	53.54
				$\boldsymbol{SDRL_1}$	496.45	493.13	489.83	483.33	467.55	410.72	270.73	53.04
				$\boldsymbol{MRL_1}$	344.11	341.81	339.53	335.02	324.08	284.69	187.65	36.76
	EEWMA	$\boldsymbol{\lambda_2 = 0.2\lambda_1}$	0.1593137	$\boldsymbol{ARL_1}$	253.55	170.07	128.07	85.91	47.43	17.55	6.13	2.07
				$\boldsymbol{SDRL_1}$	253.05	169.56	127.57	85.41	46.93	17.04	5.61	1.49
				$\boldsymbol{MRL_1}$	175.40	117.53	88.43	59.20	32.53	11.81	3.89	1.05
		$\boldsymbol{\lambda_2 = 0.6\lambda_1}$	0.1965033	$\boldsymbol{ARL_1}$	262.77	178.44	135.22	91.28	50.69	18.83	6.58	2.20
				$\boldsymbol{SDRL_1}$	262.27	177.94	134.72	90.78	50.19	18.32	6.06	1.63
				$\boldsymbol{MRL_1}$	181.79	123.34	93.38	62.92	34.79	12.70	4.20	1.15
0.15	DEWMA	$\boldsymbol{\lambda_2 = 4\lambda_1}$	0.0830442	$\boldsymbol{ARL_1}$	246.81	164.04	122.97	82.12	45.16	16.66	5.82	1.99
				$\boldsymbol{SDRL_1}$	246.30	163.54	122.47	81.62	44.66	16.15	5.30	1.40
				$\boldsymbol{MRL_1}$	170.73	113.35	84.89	56.57	30.95	11.20	3.68	0.99
		$\boldsymbol{\lambda_2 = 1.6\lambda_1}$	0.02941417	$\boldsymbol{ARL_1}$	238.73	157.02	117.12	77.82	42.61	15.69	5.50	1.90
				$\boldsymbol{SDRL_1}$	238.23	156.52	116.61	77.32	42.11	15.18	4.97	1.31
				$\boldsymbol{MRL_1}$	165.13	108.49	80.83	53.59	29.19	10.52	3.45	0.93
		$\boldsymbol{\lambda_2 = \lambda_1}$	0.01638675	$\boldsymbol{ARL_1}$	231.17	150.56	111.76	73.92	40.32	14.82	5.21	1.83
				$\boldsymbol{SDRL_1}$	230.67	150.06	111.26	73.42	39.82	14.31	4.68	1.23
				$\boldsymbol{MRL_1}$	159.89	104.01	77.12	50.89	27.60	9.92	3.25	0.88
		$\boldsymbol{\lambda_2 = 0.6\lambda_1}$	0.004486612	$\boldsymbol{ARL_1}$	***204.13***	***128.49***	***93.88***	***61.18***	***32.97***	***12.07***	***4.30***	***1.61***
				$\boldsymbol{SDRL_1}$	***203.63***	***127.99***	***93.38***	***60.67***	***32.47***	***11.55***	***3.77***	***0.99***
				$\boldsymbol{MRL_1}$	***141.15***	***88.71***	***64.72***	***42.06***	***22.51***	***8.01***	***2.62***	***0.71***

4.2. Performance Evaluation of Real-World Data for the Control Chart

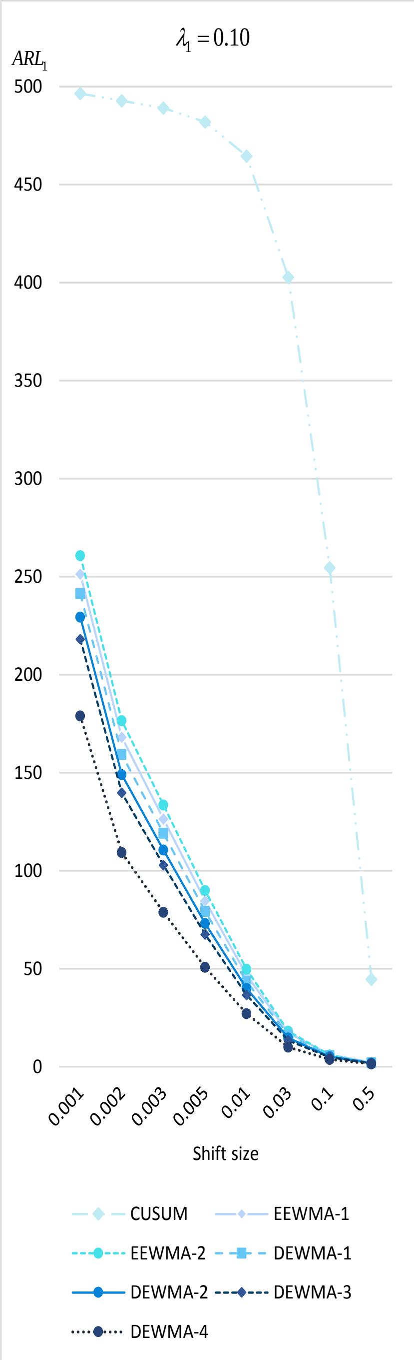
Since Thailand's economic landscape is heavily influenced by natural gas imports, which serve as a primary energy source across sectors such as power generation, manufacturing, and transportation. With the decline of domestic natural gas reserves, the nation increasingly depends on imported gas, underscoring its vital role in maintaining energy security and economic resilience. Variations in the volume and price of these imports can significantly impact energy expenses, industrial productivity, and the broader economy. Although there may have been interventions or known events in the natural gas import data—such as policy changes or market shocks—that could potentially affect the process mean and control limits, all control charts applied in this study used the same average value to compute control limits. Therefore, such factors are unlikely to bias the comparative evaluation of chart performance. Moreover, the dataset was analyzed using statistical software to identify a suitable time series model. It was found that the data follow a quadratic trend AR(p) structure, the details of which will be elaborated in the following step. This modeling process ensured that the analysis was appropriately aligned with the scope of the study. Therefore, to evaluate how the economy is doing, this study uses data on natural gas imports in Thailand, measured in units of 100 MMSCFD with a heat value of 1,000 BTU/SCF. The dataset comprises 132 monthly observations from January 2012 to December 2022, obtained from the Energy Policy and Planning Office, Ministry of Energy, Thailand. The sources of the dataset for Figures 2 and 3 are derived from the website *https://www.eppo.go.th/index.php/en/en-energystatistics/ngv-statistic*.

This dataset aligns with the model by applying time series forecasting techniques to identify the most suitable model for the data. The results indicate that the dataset follows a quadratic trend AR(p) model, which will be described in detail later. The model's suitability was assessed using SPSS software, which was employed to fit the models. Table 5 presents the coefficients for the quadratic trend AR(p) models of order 1 and 2, based on the Thailand natural gas imports dataset. Table 6 shows the accuracy values of the model fitting using MAPE and the Normalized BIC. For both criteria, lower values indicate a better model fit. The results reveal that the quadratic trend AR(2) model has lower MAPE (13.538) and Normalized BIC (1.726) compared to the AR(1) model, which yields MAPE of 14.544 and Normalized BIC of 1.845. This suggests that the AR(2) model provides a more accurate fit and is more suitable for application in this research. It is noted that the lowest MAPE and Normalized BIC values are highlighted in bold. After that, data on natural gas imports in Thailand was used to apply the AR(2) model with a quadratic trend to express the efficiency of the control chart. The next step involved using the one-sample Kolmogorov-Smirnov test to evaluate how well the white noise fits an exponential distribution with the estimated mean parameter, as shown in Table 7.

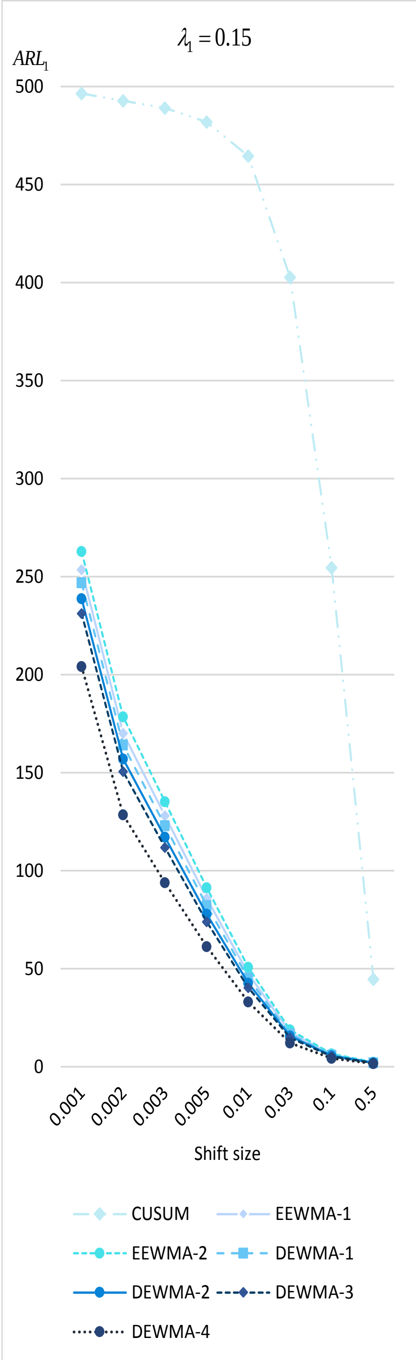
For the quadratic trend AR(2) model, the estimated exponential parameter is 1.7811, with a Kolmogorov-Smirnov statistic of 0.705 and a p-value of 0.702. Since the p-value is greater than 0.05, it indicates that the white noise does not significantly differ from the exponential distribution, confirming the appropriateness of the model. Moreover, the structure of the exponential white noise was evaluated using SPSS to verify that it met the underlying assumptions, and the analysis confirmed that these assumptions were fulfilled. Therefore, the dataset is appropriate for the quadratic trend AR(2) model and exhibits the correct parameters, which are shown to be fitted to the model as: $X_t = 0.332t - 0.002t^2 + 0.454X_{t-1} + 0.409X_{t-2} + \ldots + \phi_p X_{t-p} + \xi_t; \xi_t \sim Exp(\gamma_0 = 1.7811)$.



(a)

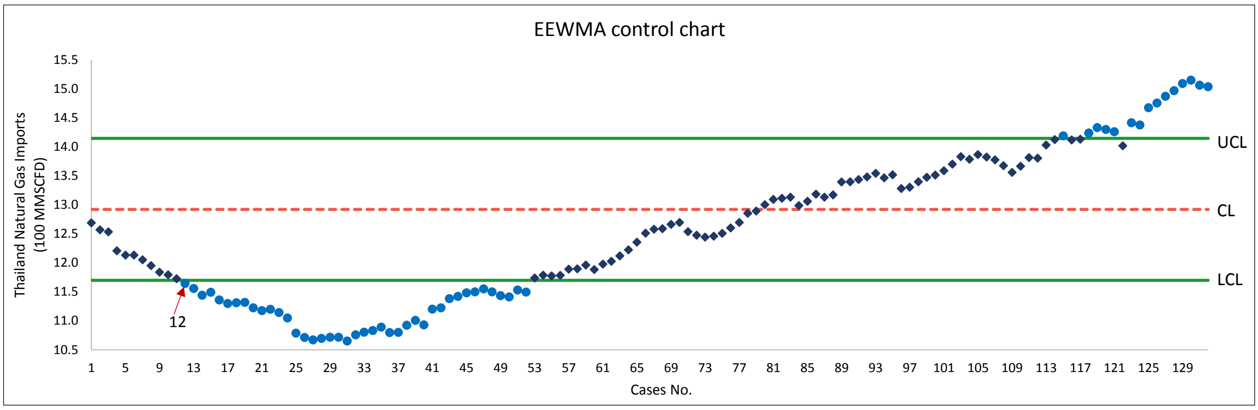


(b)

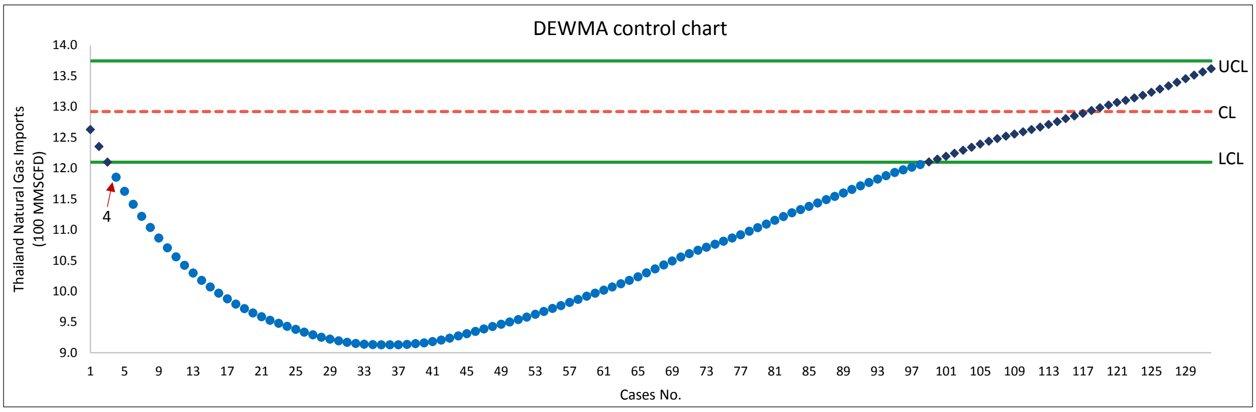


(c)

Figure 2. ARL_1 values with the Thailand natural gas imports dataset given λ_1 as; (a) 0.05, (b) 0.10 (c) 0.15



(a)



(b)

Figure 3.The capability of detecting processes of two-sided control charts of Thailand natural gas imports dataset with quadratic trend AR(2); on (a) EEWMA control chart with $\lambda_2 = 0.2\lambda_1$ and (b) DEWMA control chart with $\lambda_2 = 0.6\lambda_1$

Table 6. The coefficients for the quadratic trend AR(p) models using the Thailand natural gas imports dataset

	Quadratic trend AR(1) model				Quadratic trend AR(2) model			
Variable	**Coefficient**	**Std. Error**	**t-Statistic**	**p-value**	**Coefficient**	**Std. Error**	**t-Statistic**	**p-value**
η	0.342	0.041	8.246	0.000	0.332	0.065	5.126	0.000
ϑ	-0.002	0.000	-4.631	0.000	-0.002	0.001	-2.925	0.004
AR(1)	0.752	0.058	13.048	0.000	0.454	0.081	5.636	0.000
AR(2)					0.409	0.081	5.065	0.000

Table 7. Model Fi

Model	MAPE	Normalized BIC
Quadratic trend AR(1)	14.544	1.845
Quadratic trend AR(2)	**13.538**	**1.726**

Table 8.One-sample Kolmogorov test for the real-world data using the Thailand natural gas imports

Model	Exponential parameter (γ_0)	One-sample Kolmogorov-Smirnov	p-value
Quadratic trend AR(2)	1.7811	0.705	0.702

Table 5 presents the ARL_1, $SDRL_1$, and MRL_1values of the CUSUM, EEWMA, and DEWMA control charts under various scenarios based on the quadratic trend AR(2) model, with corresponding visualizations provided in Figure. 2. The findings reveal that decreasing the value of λ_1 results in lower ARL_1, $SDRL_1$, and MRL_1 values. The DEWMA chart consistently demonstrates superior performance compared to the EEWMA and CUSUM charts in identifying small shifts in the process mean. Furthermore, when λ_2 is set closer to λ_1, both the EEWMA and DEWMA charts exhibit improved sensitivity, as reflected in reduced ARL_1, $SDRL_1$, and MRL_1 values. This implies that selecting smoothing parameters with minimal difference between λ_1 and$\lambda_2$enhances the ability of these charts to promptly detect small process changes. The results closely align with those obtained from the simulated dataset under all conditions. Accordingly, Figure 3 illustrates the performance of the control charts in detecting process shifts during monitoring, based on the dataset by plotting the control chart graphs. The results show that, the DEWMA control chart (with $\boldsymbol{\lambda_2 = 0.6\lambda_1}$), developed using the quadratic trend AR(2) model, signalled the first out-of-control condition at the 4th observation, whereas the EEWMA control chart (with $\lambda_2 = 0.2\lambda_1$**)** did so at the 12th observation.

The results of this study highlight the superior responsiveness of the DEWMA control chart in detecting small shifts more promptly than the EEWMA control chart, especially when dealing with data exhibiting autocorrelation. In comparison with previous research, such as the EEWMA control chart under quadratic trend AR(p) [9], the adjusted MEWMA chart for linear and quadratic trend AR(p) models [20]. Notably, the findings of this study are consistent with prior research published in 2024, which enhanced the performance of the Adjusted Modified EWMA (AMEWMA) control chart for both trend and quadratic trend AR models, as well as the application of the DEWMA chart to the quadratic trend AR(1) process. Both studies also demonstrated the effectiveness of these control chart approaches when applied to economic data. It further incorporates a quadratic trend structure and exponential white noise, which had not been previously explored for the DEWMA chart. The findings from the exact ARL formula demonstrate that the enhanced DEWMA chart detects shifts more quickly, with greater accuracy and reduced computation time. This makes it a highly effective tool for practical applications in systems characterized by autocorrelated data and underlying trends such as quadratic trend.

5. Conclusion

The DEWMA chart, based on a quadratic trend AR(p) model with exponential white noise, was evaluated using the exact ARL solution, which proved more computationally efficient than the NIE method. While both approaches yield similar ARL accuracy, the exact solution offers faster performance, making it ideal for real-time or large-scale applications where quick detection of shifts is essential. Subsequently, the exact ARL solution applied to the DEWMA chart was compared with the EEWMA and CUSUM charts under out-of-control conditions with varying shift magnitudes. The comparison was conducted using ARL_1, $SDRL_1$, and MRL_1 metrics to assess detection performance. The results indicate that the DEWMA chart performed the best, particularly when λ_1 was small and $\lambda_2$was near λ_1, showing enhanced sensitivity in detecting changes to the process mean. Furthermore, these formulas can be applied to analyse real-world data, such as the natural gas import data in Thailand, which follows the AR(p) model with quadratic trend and exponential white noise. The exact solution has proven to be an effective approach for determining the ARL for shift changes observed in the DEWMA chart. By utilizing this precise *ARL* solution and evaluating the performance of the control chart with metrics such as *SDRL* and *MRL*, the sensitivity of the DEWMA chart for detecting parameter shifts was significantly improved. This enhancement contributes to improved performance in monitoring and detecting process shifts. Nonetheless, the present research provides a strong foundation for future developments aimed at increasing the sensitivity of detecting small changes across diverse data structures. While the proposed exact solution has demonstrated effectiveness, its applicability may be limited to datasets that exhibit autocorrelation and follow an autoregressive (AR) model with a quadratic trend component. Future research could focus on extending this solution to accommodate a wider variety of data types with different characteristics.

6. Declarations

6.1. Author Contributions

Conceptualization, Y.A. and K.K.; methodology, Y.A. and K.K.; formal analysis, K.K.; investigation, K.K.; data curation, K.K.; writing—original draft preparation, Y.A. and K.K.; writing—review and editing, Y.A. and K.K.; visualization, K.K. All authors have read and agreed to the published version of the manuscript.

6.2. Data Availability Statement

The data presented in this study are available in the article.

6.3. Funding and Acknowledgments

The authors were funded by Faculty of Science, Naresuan University, Thailand.

6.4. Institutional Review Board Statement

Not applicable.

6.5. Informed Consent Statement

Not applicable.

6.6. Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

7. References

[1] Page, E. S. (1954). Continuous Inspection Schemes. Biometrika, 41(1–2), 100–115. doi:10.1093/biomet/41.1-2.100.

[2] Roberts, S. W. (1959). Control Chart Tests Based on Geometric Moving Averages. Technometrics, 1(3), 239–250. doi:10.1080/00401706.1959.10489860.

[3] Khan, N., Aslam, M., & Jun, C. (2016). Design of a Control Chart Using a Modified EWMA Statistic. Quality and Reliability Engineering International, 33(5), 1095–1104. doi:10.1002/qre.2102.

[4] Naveed, M., Azam, M., Khan, N., & Aslam, M. (2018). Design of a Control Chart Using Extended EWMA Statistic. Technologies, 6(4), 108–122. doi:10.3390/technologies6040108.

[5] Shamma, S. E., & Shamma, A. K. (1992). Development and Evaluation of Control Charts Using Double Exponentially Weighted Moving Averages. International Journal of Quality & Reliability Management, 9(6), 18–25. doi:10.1108/02656719210018570.

[6] Mahmoud, M. A., & Woodall, W. H. (2010). An Evaluation of the double exponentially weighted moving average control chart. Communications in Statistics: Simulation and Computation, 39(5), 933–949. doi:10.1080/03610911003663907.

[7] Karaoglan, A. D., & Bayhan, G. M. (2012). ARL performance of residual control charts for trend AR(1) process: A case study on peroxide values of stored vegetable oil. Scientific Research and Essays, 7(13), 1405–1414. doi:10.5897/sre11.1801.

[8] Yue, S., & Pilon, P. (2003). Interaction between deterministic trend and autoregressive process. Water Resources Research, 39(4), 1210. doi:10.1029/2001wr001210.

[9] Karoon, K., Areepong, Y., & Sukparungsee, S. (2023). On the Performance of the Extended EWMA Control Chart for Monitoring Process Mean Based on Autocorrelated Data. Applied Science and Engineering Progress, 16(4), 6599. doi:10.14416/j.asep.2023.01.004.

[10] Champ, C. W., & Ritrdon, S. E. (1991). A Comparison of the Markov Chain and the Integral Equation Approaches for Evaluating the Run Length Distribution of Quality Control Charts. Communications in Statistics - Simulation and Computation, 20(1), 191–204. doi:10.1080/03610919108812948.

[11] Riaz, M., Zaman, B., Raji, I. A., Omar, M. H., Mehmood, R., & Abbas, N. (2022). An Adaptive EWMA Control Chart Based on Principal Component Method to Monitor Process Mean Vector. Mathematics, 10(12), 27. doi:10.3390/math10122025.

[12] Peerajit, W. (2021). Cumulative Sum Control Chart Applied to Monitor Shifts in the Mean of a Long-memory ARFIMAX(p,d*,q,r) Process with Exponential White Noise. Thailand Statistician, 20(1), 144–161.

[13] Supharakonsakun, Y. (2021). Statistical design for monitoring process mean of a modified EWMA control chart based on autocorrelated data. Walailak Journal of Science and Technology, 18(12), 19183. doi:10.48048/wjst.2021.19813.

[14] Bualuang, D., & Peerajit, W. (2023). Performance of the CUSUM Control Chart Using Approximation to ARL for Long-Memory Fractionally Integrated Autoregressive Process with Exogenous Variable. Applied Science and Engineering Progress, 16(2), 5917.

[15] Karoon, K., & Areepong, Y. (2023). Improving Sensitivity of the DEWMA Chart with Exact ARL Solution under the Trend AR(p) Model and Its Applications. Emerging Science Journal, 7(6), 1875–1891. doi:10.28991/esj-2023-07-06-03.

[16] Phanyaem, S. (2024). Precise Average Run Length of an Exponentially Weighted Moving Average Control Chart for Time Series Model. Thailand Statistician, 22(4), 909–925.

[17] Phanyaem, S. (2024). Development of average run length formulas for EWMA control chart under the AR(1) with quadratic trend model for detecting and monitoring process variability. Journal of Applied Science and Emerging Technology, 23(2), e257034. doi:10.14416/JASET.KMUTNB.2024.02.004.

[18] Karoon, K., & Areepong, Y. (2025). The Efficiency of the New Extended EWMA Control Chart for Detecting Changes Under an Autoregressive Model and Its Application. Symmetry, 17(1), 104. doi:10.3390/sym17010104.

[19] Neammai, J., Sukparungsee, S., & Areepong, Y. (2025). Explicit Analytical Form for the Average Run Length of Double-Modified Exponentially Weighted Moving Average Control Charts Through the MA(q) Process and Applications. Symmetry, 17(2), 238. doi:10.3390/sym17020238.

[20] Karoon, K., & Areepong, Y. (2024). Enhancing the Performance of an Adjusted MEWMA Control Chart Running on Trend and Quadratic Trend Autoregressive Models. Lobachevskii Journal of Mathematics, 45(4), 1601-1617. doi:10.1134/S1995080224601577.

[21] Almousa, M. (2020). Adomian decomposition method with modified bernstein polynomials for solving nonlinear fredholm and volterra integral equations. Mathematics and Statistics, 8(3), 278–285. doi:10.13189/ms.2020.080305.

[22] Naveed, M., Azam, M., Khan, N., Aslam, M., & Albassam, M. (2021). Designing of control chart of extended EWMA statistic using repetitive sampling scheme. Ain Shams Engineering Journal, 12(1), 1049-1058. doi:10.1016/j.asej.2020.09.004.

[23] Shukla, S., Balasubramanian, S., & Pavlović, M. (2016). A Generalized Banach Fixed Point Theorem. Bulletin of the Malaysian Mathematical Sciences Society, 39(4), 1529–1539. doi:10.1007/s40840-015-0255-5.

[24] Sofonea, M., Han, W., & Shillor, M. (2005). Analysis and Approximation of Contact Problems with Adhesion or Damage. Chapman and Hall/CRC, New York, United States. doi:10.1201/9781420034837.