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Relativistic Extended Thermodynamics of Polyatomic Gases with Rotational and Vibrational Modes

Sebastiano Pennisi [a*]

[a] *Department of Mathematics and Informatics, University of Cagliari, Cagliari, Italy.*

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Abstract

In a recent article, an infinite set of balance equations has been proposed to modelize polyatomic gases with rotational and vibrational modes in a non-relativistic context. To obtain particular cases, it has been truncated to obtain a model with 7 or 15 moments. Here the following objectives are pursued: 1) to obtain the relativistic counterpart of this model, which, at the non-relativistic limit, gives the same balance equations as in the known classical case; 2) to obtain the previous result for the model with an arbitrary but fixed number of moments; and 3) to obtain the closure of the resulting relativistic model so that all the functions appearing in the balance equations are expressed in terms of the independent variables. To achieve these goals, the following methods are used: 1) the principle of entropy is imposed. As a result, it is obtained that the closure is determined up to a single 4-vectorial function, usually called a 4-potential. 2) To determine this last function, a more restrictive principle is imposed, namely the Maximum Entropy Principle (MEP). 3) Since all the functions involved must be expressed in the covariant form so as not to depend on the observer, the Representation Theorems are used. The findings of this article exactly match the goals outlined earlier. They are clearly novel because they have never been achieved before. They can also be considered improvements because, if the aforementioned arbitrary number of moments is restricted to 16, the present work coincides with that already known in literature.

Keywords: Moments Equations; Extended Thermodynamics; Non-equilibrium Thermodynamics..

1. Introduction

Based on Arima et al. (2018) [1] study (recently improved to describe dense polyatomic gases in Arima et al. (2020) [2]), the following balance equations have been introduced:

$$\begin{aligned}
&\partial_t F^{i_1\cdots i_r} + \partial_k F^{k i_1\cdots i_r} = P^{i_1\cdots i_r}\,,\\
&\partial_t F_V^{i_1\cdots i_r} + \partial_k F_V^{k i_1\cdots i_r} = P_V^{i_1\cdots i_r}\,,\\
&\partial_t F_R^{i_1\cdots i_r} + \partial_k F_R^{k i_1\cdots i_r} = P_R^{i_1\cdots i_r}\,,
\end{aligned} \tag{1}$$

Where r goes from 0 to $+\infty$,

$$\begin{aligned}
&F^{i_1\cdots i_r} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^{i_1}\cdots\xi^{i_r}\phi(\mathcal{J}^R)\,\psi(\mathcal{J}^V)\,d\,\mathcal{J}^R\,d\,\mathcal{J}^V\,d\,\vec{\xi}\,,\\
&F_V^{i_1\cdots i_r} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^{i_1}\cdots\xi^{i_r}\,\frac{2\,\mathcal{J}^V}{m}\,\phi(\mathcal{J}^R)\,\psi(\mathcal{J}^V)\,d\,\mathcal{J}^R\,d\,\mathcal{J}^V\,d\,\vec{\xi}\,,\\
&F_R^{i_1\cdots i_r} = m\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\,\xi^{i_1}\cdots\xi^{i_r}\,\frac{2\,\mathcal{J}^R}{m}\,\phi(\mathcal{J}^R)\,\psi(\mathcal{J}^V)\,d\,\mathcal{J}^R\,d\,\mathcal{J}^V\,d\,\vec{\xi}\,.
\end{aligned} \tag{2}$$

* Corresponding author: spennisi@unica.it

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The definitions of $F^{ki_1\cdots i_r}$, $F_V^{ki_1\cdots i_r}$, $F_R^{ki_1\cdots i_r}$ are similar, with an extra factor ξ^k inside the integrals. Moreover, they have $P = 0$, $P^i = 0$, $P_V + P_R + P^{ll} = 0$ in order to ensure the conservation laws of mass, momentum and total energy. After that, the truncated systems with 7 and 15 moments have been considered and fully investigated. The Equations (1) $_1$ have been called the mass block, while (1) $_{2,3}$ constitute the vibrational and rotational blocks respectively. In the previous models for monoatomic gases, only the mass block (1) $_1$ was considered (see for example Liu and Müller (1983) [3], Müller, T. Ruggeri (1998) [4]). Its extension to the polyatomic gases began with Arima et al. (2012) [5] and gave inspiration, as the previous one, to many other articles part of which are cited in Ruggeri and Sugiyama (2015) [6]. In these articles, the two blocks of Equations (1) $_{1,2}$ were considered. The subsequent generalization [1], considers all the three blocks of Equations (1) $_{1-3}$ by distinguishing the contribute of rotational and vibrational modes. The sum of (1) $_2$, (1) $_3$ and of the trace of (1) $_1$ can substitute (1) $_3$ and reads:

$$\partial_t H_0^{i_1\cdots i_r} + \partial_k H_0^{ki_1\cdots i_r} = J_0^{i_1\cdots i_r}\,, \tag{3}$$

With:

$$H_0^{i_1\cdots i_r} = F_V^{i_1\cdots i_r} + F_R^{i_1\cdots i_r} + F^{i_1\cdots i_{r+2}}\delta_{i_{r+1}i_{r+2}}\,,$$
$$H_0^{ki_1\cdots i_r} = F_V^{ki_1\cdots i_r} + F_R^{ki_1\cdots i_r} + F^{ki_1\cdots i_{r+2}}\delta_{i_{r+1}i_{r+2}}\,,$$
$$J_0^{i_1\cdots i_r} = P_V^{i_1\cdots i_r} + P_R^{i_1\cdots i_r} + P^{i_1\cdots i_{r+2}}\delta_{i_{r+1}i_{r+2}}\,.$$

In the sequel we will use also the quantities;

$$\begin{aligned} &H_q^{i_1\cdots i_r} = H_0^{i_1\cdots i_r l_1 l_1\cdots l_{q-1}l_{q-1}}\,, \quad H_q^{ki_1\cdots i_r} = H_0^{ki_1\cdots i_r l_1 l_1\cdots l_{q-1}l_{q-1}} \\ &\widetilde{H}_q^{i_1\cdots i_r} = F_V^{i_1\cdots i_r l_1 l_1\cdots l_{q-1}l_{q-1}} + F_R^{i_1\cdots i_r l_1 l_1\cdots l_{q-1}l_{q-1}}\,, \\ &\widetilde{H}_q^{ki_1\cdots i_r} = F_V^{ki_1\cdots i_r l_1 l_1\cdots l_{q-1}l_{q-1}} + F_R^{ki_1\cdots i_r l_1 l_1\cdots l_{q-1}l_{q-1}}\,. \end{aligned} \tag{4}$$

Obviously, from Equations (3) and (1) $_{2,3}$ it follows:

$$\partial_t H_q^{i_1\cdots i_r} + \partial_k H_q^{ki_1\cdots i_r} = \overline{P}^{i_1\cdots i_r}\,, \quad \partial_t \widetilde{H}_q^{i_1\cdots i_r} + \partial_k \widetilde{H}_q^{ki_1\cdots i_r} = \tilde{P}^{i_1\cdots i_r}\,, \tag{5}$$

With obvious meaning of $\overline{P}^{i_1\cdots i_r}$ and $\tilde{P}^{i_1\cdots i_r}$.

Here we propose to find the relativistic counterpart of (1); since this non relativistic approach started from the classical Boltzman-Chernikov equation, we do the same starting from the generalized relativistic Boltzman-Chernikov equation;

$$p^\alpha\, \partial_\alpha \alpha\, f = Q\,,$$

where f is the distribution function. By multipying it by polynomials p in the 4-momentum p^α, by a function f_1 of the rotational energy $\mathcal{I}^R$, by a function f_2 of the vibrational energy $\mathcal{I}^v$, by the product of their measures $\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)$ and integrating in $d\,\mathcal{I}^R\,d\,\mathcal{I}^V\,d\,\vec{P}$, one obtains a field equations. So the problem is now how to determine these quantities p, f_1 and f_2 such that the resulting relativistic field equations have (1) as non reltivistic limit. We have found the result expressed by the following set of balance equations as relativistic counterpart of (1) truncated in a convenient way in terms of an arbitrary but fixed integer non negative number S:

$$\partial_\alpha A^{\alpha\alpha_1\cdots\alpha_r} = I^{\alpha_1\cdots\alpha_r}\,, \quad \partial_\alpha A_V^{\alpha\alpha_1\cdots\alpha_s} = I_V^{\alpha_1\cdots\alpha_s}\,, \tag{6}$$

With $r = 0\,,\cdots\,,S+2$, $s = 0\,,\cdots\,,S$ and where;

$$\begin{aligned} &A^{\alpha\alpha_1\cdots\alpha_r} = \frac{c}{m^{r-1}}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\, p^\alpha p^{\alpha_1}\cdots p^{\alpha_r}\left(1+\frac{r\,\mathcal{I}}{m\,c^2}\right)\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\,\mathcal{I}^R\,d\,\mathcal{I}^V\,d\,\vec{P}\,, \\ &A_V^{\alpha\alpha_1\cdots\alpha_s} = \frac{c}{m^{s-1}}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\, p^\alpha p^{\alpha_1}\cdots p^{\alpha_s}\frac{2\,\mathcal{I}^V}{m\,c^2}\,\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\,\mathcal{I}^R\,d\,\mathcal{I}^V\,d\,\vec{P}\,, \end{aligned} \tag{7}$$

and $\mathcal{I} = \mathcal{I}^R + \mathcal{I}^V$.

Despite the appearances, in the system (6) there is a complete symmetry between the rotational and vibrational modes. In fact, for $r = 0\,,\cdots\,,S$ we can add to each equation of (6) $_1$ the trace of that with $r+2$ instead of r multiplied by $-\,c^{-2}$; after that, we sum the corresponding equation in (6) $_2$ so obtaining;

$$\partial_\alpha A_R^{\alpha\alpha_1\cdots\alpha_r} = I_R^{\alpha_1\cdots\alpha_r}\,, \quad with \quad I_R^{\alpha_1\cdots\alpha_r} = -I^{\alpha_1\cdots\alpha_r} + c^{-2} I_R^{\alpha_1\cdots\alpha_{r+2}} g_{\alpha_{r+1}\alpha_{r+2}} - I_V^{\alpha_1\cdots\alpha_r}\,,$$

And $A_R^{\alpha\alpha_1\cdots\alpha_r}$ defined as (7) $_2$ with R instead of V except that in $\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)$. This is a consequence of the property $p^\alpha p_\alpha = m^2c^2$. It follows that the system (6) can be written also as;

$$\begin{aligned} &\partial_\alpha A_R^{\alpha\alpha_1\cdots\alpha_r} = I_R^{\alpha_1\cdots\alpha_r}\,, \quad \partial_\alpha A_V^{\alpha\alpha_1\cdots\alpha_r} = I_V^{\alpha_1\cdots\alpha_r}\,, \quad with \quad r = 0\,,\cdots\,,S\,, \\ &\partial_\alpha A^{\alpha\alpha_1\cdots\alpha_{S+1}} = I^{\alpha_1\cdots\alpha_{S+1}}\,, \quad \partial_\alpha A^{\alpha\alpha_1\cdots\alpha_{S+2}} = I^{\alpha_1\cdots\alpha_{S+2}}\,. \end{aligned} \tag{8}$$

We note that, from the definition $(7)_2$ it follows that the trace conditions hold;

$$A_R^{\alpha\alpha_1\cdots\alpha_r} g_{\alpha_{r-1}\alpha_r} = c^2 A_R^{\alpha\alpha_1\cdots\alpha_{r-2}}\,, \quad A_V^{\alpha\alpha_1\cdots\alpha_r} g_{\alpha_{r-1}\alpha_r} = c^2 A_V^{\alpha\alpha_1\cdots\alpha_{r-2}}\,. \tag{9}$$

In the next section we will find the non relativistic limit of Equations (6) and the resulting field equations are reported in the subsequent Equations (10). By comparing them with the above Equations (1), the following facts become evident:

- The mass block $(1)_1$ has to be considered for $0 \leq r \leq S+2$.
- The vibrational and rotational blocks appear for $0 \leq r \leq S$; also the subsequent orders $S+1, S+2, \cdots, 2S$ have to be considered (here "order" of a tensor is the number of its indexes) but only by means of their traces and this is according to the law: "If r is the number of its free indexes, then $S-r$ traces have to be taken, for $0 \leq r \leq S-1$".
- A number of traces, less than $S-r$, have also to be considered but only by means of the sum of the tensors in the rotational and that in the vibrational mode. In fact, from $(10)_7$ we see that $q \leq S-r$ and consequently, from (13) we see that the number of traces there occurring is $q-1 \leq S-r-1$.
- Equations involving terms of the mass block $(1)_1$ of order $S+3, S+4, \cdots, 2S+4$ have to be considered but only by means of the sum of them and that belonging to the rotational and vibrational blocks. In fact, from $(4)_1$ and $(3)_3$ we see that $H_q^{i_1\cdots i_r}$ involves the tensor of the mass block of order $2q+r$; from $(10)_6$ we see that $H_q^{i_1\cdots i_{S-q+2}}$ involves a tensor of order $S+q+2$ and we see also that $S+3 \leq S+q+2 \leq 2S+4$. However, this tensor appears only after having taken its trace q times.

We note that the model introduced in Pennisi and Ruggeri (2020) [7] is a particular case of the present one; in fact, with the theory of subsystems developed in Boillat and Ruggeri (1997) [8], by dropping out $(6)_2$, what remains gives the model of Pennisi and Ruggeri (2020) [7] were there was considered no distinction between the rotational and vibrational modes. Moreover, the present model has been tested in the simpler case $S=0$ and the results have been published in [9]; this correspondence will be verified in Sect. 3. So also the model in Carrisi and Pennisi (2019) [9] is a subsystem of the present one when $S=0$.

In section 5 we will find the closure of the new field Equations (6). It is expressed by the Equations (33) jointly with (23) reported below in that section. In this way the first parts of the following flowchart have been described. In particular, in this introduction its first step was obtained, i.e., the field Equations (6) as relativistic counterpart of the classical model (1) proposed in Pennisi and Ruggeri (2017) [11]. Flowchart of the research methodology is presented by Figure 1.

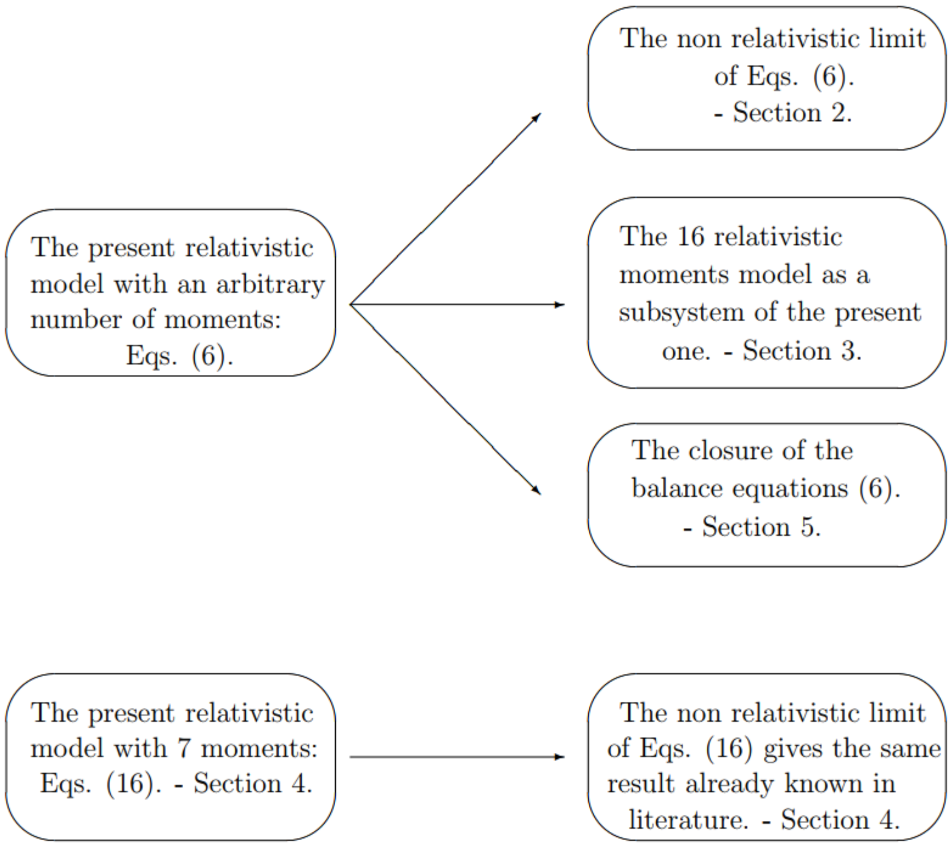

Figure 1. Flowchart of the research methodology

2. The Non-relativistic Limit

We prove now that the non relativistic limit of Equations (6) leads to the following hierarchy of balance equations for the classical case:

$$
\begin{aligned}
&\partial_t F^{i_1\cdots i_r} + \partial_k F^{k i_1\cdots i_r} = P^{i_1\cdots i_r}\,, \quad for \quad 0 \le r \le S+2\,, \\
&\left.\begin{aligned}
&\partial_t F_V^{i_1\cdots i_r} + \partial_k F_V^{k i_1\cdots i_r} = P_V^{i_1\cdots i_r}\,, \\
&\partial_t F_R^{i_1\cdots i_r} + \partial_k F_R^{k i_1\cdots i_r} = P_R^{i_1\cdots i_r}
\end{aligned}\right. \quad , \quad for \quad 0 \le r \le S\,, \\
&\left.\begin{aligned}
&\partial_t F_V^{i_1\cdots i_r l_1 l_1 \cdots l_{S-r} l_{S-r}} + \partial_k F_V^{k i_1\cdots i_r l_1 l_1 \cdots l_{S-r} l_{S-r}} = Q_V^{i_1\cdots i_r}\,, \\
&\partial_t F_R^{i_1\cdots i_r l_1 l_1 \cdots l_{S-r} l_{S-r}} + \partial_k F_R^{k i_1\cdots i_r l_1 l_1 \cdots l_{S-r} l_{S-r}} = Q_R^{i_1\cdots i_r}\,,
\end{aligned}\right. \quad for \quad 0 \le r \le S-1\,, \\
&\partial_t H_q^{i_1\cdots i_{S-q+2}} + \partial_k H_q^{k i_1\cdots i_{S-q+2}} = J_q^{i_1\cdots i_{S-q+2}}\,, \quad for \quad 1 \le q \le S+2\,, \\
&\partial_t \widetilde{H}_q^{i_1\cdots i_r} + \partial_k \widetilde{H}_q^{k i_1\cdots i_r} = \widetilde{J}_q^{i_1\cdots i_r}\,, \quad for \quad \begin{cases} 2 \le q \le S\,, \\ 0 \le r \le S-q\,, \end{cases}
\end{aligned}
\tag{10}
$$

With $H_q^{i_1\cdots i_r}$ and $\widetilde{H}_q^{i_1\cdots i_r}$ defined below in Equations (12) and (13).

This result have already been described at the end of the previous section. So there remains here to prove it. Thanks to the trace condition (9) $_2$ and to (6) $_2$, we see that we can applicate the results of Borghero et al. (2005) [10]. We have only to observe that in this article there are 2 arbitrary numbers $M < N$. We observe also that the free indexes appearing in Equation (1) of Borghero et al. (2005) [10] starts from α_2 instead of α_1 as in the present article. So by comparing these equations with the present (6) $_2$, we see that $N = S+1$, $M = S$. After that, we can use Equation (2) of Borghero et al. (2005) [10] and see that the non relativistic limit of the present Equation (6) $_2$ gives the above reported Equations (10) $_{2,4}$. Let us consider now the present Equation (6) $_1$; there isn't a trace condition on it, so that we cannot apply the results of Borghero et al. (2005) [10]. But we can apply those in Equation (11) of Pennisi and Ruggeri (2020) [7] and have that its non relativistic limit is:

$$
\partial_t H_q^{i_1\cdots i_r} + \partial_k H_q^{k i_1\cdots i_r} = J_q^{i_1\cdots i_r}\,, \quad for \quad \begin{cases} 0 \le q \le S+2\,, \\ 0 \le r \le S-q+2\,, \end{cases}
\tag{11}
$$

$$
\begin{aligned}
H_q^{i_1\cdots i_r} &= m \int_{\mathbb{R}^3} \int_0^{+\infty} \int_0^{+\infty} f^C\, \xi^{i_1} \cdots \xi^{i_r}\, \xi^{2(q-1)} \left(\xi^2 \; + \; 2q \frac{\mathcal{I}}{m}\right) \phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\,\mathcal{I}^R\, d\,\mathcal{I}^V\, d\,\vec{\xi}\,, \\
H_q^{k i_1\cdots i_r} &= m \int_{\mathbb{R}^3} \int_0^{+\infty} \int_0^{+\infty} f^C\, \xi^k \xi^{i_1} \cdots \xi^{i_r}\, \xi^{2(q-1)} \left(\xi^2 \; + \; 2q \frac{\mathcal{I}}{m}\right) \phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\,\mathcal{I}^R\, d\,\mathcal{I}^V\, d\,\vec{\xi}\,.
\end{aligned}
\tag{12}
$$

We now further elaborate these last Equations (11) and (12).

First Step: For $\begin{cases} 1 \le q \le S+2\,, \\ 0 \le r \le S-q+1\,, \end{cases}$ we substitute $H_q^{i_1\cdots i_r}$ with;

$$
\begin{aligned}
\widetilde{H}_q^{i_1\cdots i_r} &= H_q^{i_1\cdots i_r} - H_{q-1}^{i_1\cdots i_{r+2}} \delta_{i_{r+1} i_{r+2}} = \\
&= m \int_{\mathbb{R}^3} \int_0^{+\infty} \int_0^{+\infty} f^C\, \xi^{i_1} \cdots \xi^{i_r}\, \xi^{2(q-1)}\, \frac{2\,\mathcal{I}}{m}\, \phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\,\mathcal{I}^R\, d\,\mathcal{I}^V\, d\,\vec{\xi}\,.
\end{aligned}
\tag{13}
$$

We can do this because both $(r\,, q)$ and $(r+2\,, q-1)$ satisfy the condition (11) $_2$. Of course, we do this starting from the highest value of q (i.e. $S+2$) to go down to $q = 1$.

So, now the equations are (10) $_{2,4}$, (11) for $q = 0$, $0 \le r \le S+2$, (11) for $\begin{cases} 1 \le q \le S+2\,, \\ r = S-q+2\,, \end{cases}$ and:

$$
\partial_t \widetilde{H}_q^{i_1\cdots i_r} + \partial_k \widetilde{H}_q^{k i_1\cdots i_r} = \widetilde{J}_q^{i_1\cdots i_r}\,, \quad for \quad \begin{cases} 1 \le q \le S+2\,, \\ 0 \le r \le S-q+1\,, \end{cases}
\tag{14}
$$

We note that (11) for $q = 0$, $0 \le r \le S+2$ are exactly the Equations (10) $_1$ of the mass block.

Second Step: Let us explicitate the subset of Equation (14) with $q = 1$, i.e.,

$$\partial_t \tilde{H}_1^{i_1\cdots i_r} + \partial_k \tilde{H}_1^{k i_1\cdots i_r} = \tilde{J}_1^{i_1\cdots i_r}\,, \quad for \quad 0 \le r \le S\,.$$

It is easy to recognize that these equations are equivalent to (10) $_3$ because from Equation (13) we desume that $\tilde{H}_1^{i_1\cdots i_r} = F_R^{i_1\cdots i_r} + F_V^{i_1\cdots i_r}$.

After that, the first condition in (14) $_2$ must be replaced by $2 \le q \le S+2$. But when $q = S+2$, the second condition in (14) $_2$ becomes $0 \le r \le -1$. It follows that the first condition in (14) $_2$ must be replaced by $2 \le q \le S+1$.

Let us explicitate now the subset of Equation (14) with $q = S - r + 1$; from the condition (14) $_2$ we see that this can be done only when $\begin{cases} 2 \le S - r + 1 \le S + 1\,, \\ 0 \le r \le r\,, \end{cases}$, i.e., $0 \le r \le S - 1$.

But, for $0 \le r \le S-1$ we have also;

$$\tilde{H}_{S-r+1}^{i_1\cdots i_r} = F_V^{i_1\cdots i_r l_1 l_1 \cdots l_{S-r} l_{S-r}} + F_R^{i_1\cdots i_r l_1 l_1 \cdots l_{S-r} l_{S-r}}\,.$$

It follows that, from each equation of the subset of Equation (14) with $q = S - r + 1$ we can subtract Equations (10) $_4$ and obtain (10) $_5$.

So, now the equations are (10) $_{1-5}$, (11) for $r = S - q + 2$, $1 \le q \le S+2$, (14) for $\begin{cases} 2 \le q \le S + 1\,, \\ 0 \le r \le S - q\,, \end{cases}$. But, for $q = S+1$ the second of these conditions becomes $0 \le r \le -1$; so the first condition must be replaced by $2 \le q \le S$. The corresponding equations are the above reported (10) $_7$, while (11) for $r = S - q + 2$, $1 \le q \le S+2$ is the above reported (10) $_6$. This completes the proof.

We conclude this section noting that, by changing index in (10) $_6$ according to the law $q = S + 2 - r$ and by taking into account (4) $_1$, it becomes;

$$\partial_t H_0^{i_1\cdots i_r l_1 l_1 \cdots l_{S+1-r} l_{S+1-r}} + \partial_k H_0^{k i_1\cdots i_r l_1 l_1 \cdots l_{S+1-r} l_{S+1-r}} = J_{S+2-r}^{i_1\cdots i_r}\,, \tag{15}$$

for $0 \le r \le S+1$.

From (3) $_2$ we see that $H_0^{i_1\cdots i_r l_1 l_1 \cdots l_{S+1-r} l_{S+1-r}}$, $H_0^{k i_1\cdots i_r l_1 l_1 \cdots l_{S+1-r} l_{S+1-r}}$ are respectively equal to $F^{i_1\cdots i_r l_1 l_1 \cdots l_{S+2-r} l_{S+2-r}}$, $F^{k i_1\cdots i_r l_1 l_1 \cdots l_{S+2-r} l_{S+2-r}}$ plus terms of the rotational and vibrational modes. So, in the subcase without these rotational and vibrational modes, Equation (15) is the counterpart of (10) $_{4,5}$ for the mass block, obviously with $S+2$ instead of S.

In this way the second step of the above flowchart has been obtained, i.e., that the non relativistic limit of the field Equations (6) gives exactly those of the classical model (1) proposed in Pennisi and Ruggeri (2017) [11]; moreover, a further information has been achieved, i.e., how the classical Equations (1) must be interrupted to obtain a model with a finite set of equations.

3. The Particular Case S=0

In this case the Equations (6) and (7) become:

$$\partial_\alpha A^\alpha = 0\,, \quad \partial_\alpha A^{\alpha\alpha_1} = 0\,, \quad \partial_\alpha A^{\alpha\alpha_1\alpha_2} = I^{\alpha_1\alpha_2}\,, \quad \partial_\alpha A_V^\alpha = I_V\,, \tag{16}$$

$$\begin{aligned}
A^\alpha &= m\,c \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, p^\alpha \phi(\mathcal{J}^R)\, \psi(\mathcal{J}^V)\, d\,\mathcal{J}^R\, d\,\mathcal{J}^V\, d\,\vec{P}\,, \\
A^{\alpha\alpha_1} &= c \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, p^\alpha p^{\alpha_1} \left(1 + \frac{\mathcal{J}}{m\,c^2}\right) \phi(\mathcal{J}^R)\, \psi(\mathcal{J}^V)\, d\,\mathcal{J}^R\, d\,\mathcal{J}^V\, d\,\vec{P}\,, \\
A^{\alpha\alpha_1\alpha_2} &= \frac{c}{m} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, p^\alpha p^{\alpha_1} p^{\alpha_2} \left(1 + \frac{2\,\mathcal{J}}{m\,c^2}\right) \phi(\mathcal{J}^R)\, \psi(\mathcal{J}^V)\, d\,\mathcal{J}^R\, d\,\mathcal{J}^V\, d\,\vec{P}\,, \\
A_V^\alpha &= m\,c \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} f\, p^\alpha \frac{2\,\mathcal{J}^V}{m\,c^2}\, \phi(\mathcal{J}^R)\, \psi(\mathcal{J}^V)\, d\,\mathcal{J}^R\, d\,\mathcal{J}^V\, d\,\vec{P}\,.
\end{aligned} \tag{17}$$

This is the 16 moments model which is present in Carrisi and Pennisi (2019) [9]. If we take off the trace of the third equation, as it was done in Pennisi and Ruggeri (2017) [11], we obtain a 15 moments model which is the relativistic counterpart of Equations (12) in Arima et al. (2018) [1]. Its non relativistic limit can be desumed from the above Equations (10).

- In particular, (10) $_1$ gives 10 equations of the mass block, i.e., (12) $_{1-3}$ of Arima et al. (2018) [1].
- Equations (10) $_{2,3}$ are to be considered only for $r = 0$ and give Equations (12) $_{4-5}$ of Arima et al. (2018) [1], i.e.,

the balance equations of the vibrational and rotational energies respectively.

- Equations (10) $_{4,5}$ aren't to be considered because they hold only for the empty set $0 \leq r \leq -1$.
- Equations (10) $_6$ have to be considered only for $q = 1$ and $q = 2$. With the first of these values we obtain (12) $_6$ of Arima et al. (2018) [1]; with $q = 2$ we obtain the hybrid equation which is present in Equations (3) $_6$ of Carrisi and Pennisi (2019) [9].
- Equation (10) $_7$ must not to be considered because it holds only for the empty set $2 \leq q \leq 0$, $0 \leq r \leq -q$.

So the third step of the above flowchart has been obtained, i.e., the subsystem with 16 moments and it is exactly the model considered in Carrisi and Pennisi (2019) [9]; moreover, if we take off the trace of the third equation, as it was done in Pennisi and Ruggeri (2017) [11], we obtain a further subsytem, i.e., the 15 moments model which is the relativistic counterpart of Equations (12) in Arima et al. (2018) [1], one of the two model with a finite number of equations proposed there.

4. The 7 Moments Model

This case is presented in subsection 3.5 of Arima et al. (2018) [1] and is described by the balance equations;

$$\begin{aligned} \partial_t F + \partial_k F^k = 0\,, \quad \partial_t F^{i_1} + \partial_k F^{ki_1} = 0\,, \quad & \partial_t F^{ll} + \partial_k F^{kll} = -P_V^{ll} - P_R^{ll}\,, \\ & \partial_t F_V + \partial_k F_V^k = P_V^{ll}\,, \\ & \partial_t F_R + \partial_k F_R^k = P_R^{ll}\,. \end{aligned} \tag{18}$$

Its relativistic counterpart cannot be written as (6) but as;

$$\partial_\alpha A^\alpha = 0\,, \quad \partial_\alpha A^{\alpha\alpha_1} = 0\,, \quad \partial_\alpha A^{\alpha\alpha_1\alpha_2} g_{\alpha_1\alpha_2} = I^{\alpha_1\alpha_2} g_{\alpha_1\alpha_2}\,, \quad \partial_\alpha A_V^\alpha = I_V\,. \tag{19}$$

In fact, $\frac{1}{c^2}\, A^{\alpha\alpha_1\alpha_2} g_{\alpha_1\alpha_2} = A^\alpha + A_R^\alpha + A_V^\alpha$ so that the third equation can be substituted by;

$$\partial_\alpha A_R^\alpha = I_R \stackrel{def}{=} \frac{1}{c^2}\, I^{\alpha_1\alpha_2} g_{\alpha_1\alpha_2} - I_V\,, \tag{20}$$

which is the counterpart of Equation (19) $_4$ with the rotational mode instead of the vibrational one.

The non relativistic limit of Equations (19) $_{1,2}$ has been calculated in Equation (17) of Pennisi and Ruggeri (2017) [11] and is;

$$\partial_t F + \partial_k F^k = 0\,, \quad \partial_t F^{i_1} + \partial_k F^{ki_1} = 0\,, \quad \partial_t G^{ll} + \partial_k G^{kll} = 0\,, \tag{21}$$

with $G^{ll} = F^{ll} + F_V^{ll} + F_R^{ll}$, $G^{kll} = F^{kll} + F_V^{kll} + F_R^{kll}$. Now, the non relativistic limits of (19) $_3$ and (20) are respectively (18) $_{4,5}$. By subtracting them from (21) $_3$, we see that (21) become (18) $_{1-3}$. This completes our proof. So the last step of the above flowchart has been obtained, i.e., the subsystem with 7 moments (19) which is the relativistic counterpart of Equations (18) which describe the second and last example with a finite number of equations proposed in Arima et al. (2018) [1].

5. The Closure of the New Relativistic Field Equations

By using the Maximum Entropy Pinciple, as in Pennisi and Ruggeri (2017) [11] and recently in Mentrelli and Ruggeri (2021) [12], we find that the distribution funcion f has the form;

$$\begin{aligned} & f = e^{-1-\frac{1}{k_B}\chi} \quad with \\ & \chi = \frac{1}{m^{r-1}}\left(1 + \frac{r\,\mathcal{I}}{m\,c^2}\right) p^{\alpha_1}\cdots p^{\alpha_r}\,\lambda_{\alpha_1\cdots\alpha_r} + \frac{1}{m^{s-1}}\left(\frac{2\,\mathcal{I}^V}{m\,c^2}\right) p^{\alpha_1}\cdots p^{\alpha_s}\,\mu_{\alpha_1\cdots\alpha_s}\,, \end{aligned}$$

Where summation over the indexes r and s is implied and k_B is the Boltzmann constant. So, if we define the 4-potential;

$$h'^\alpha = -\,k_B c \int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty} f\, p^\alpha \phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\,\mathcal{I}^R\,d\,\mathcal{I}^V\,d\,\vec{P}\,,$$

We have;

$$A^{\alpha\alpha_1\cdots\alpha_r} = \frac{\partial\, h'^\alpha}{\partial\,\lambda_{\alpha_1\cdots\alpha_r}} \quad , \quad A_V^{\alpha\alpha_1\cdots\alpha_s} = \frac{\partial\, h'^\alpha}{\partial\,\mu_{\alpha_1\cdots\alpha_s}}.$$

Since $h'^\alpha\,\xi_\alpha$ is a convex function of the Lagrange multipliers for whatever time-like 4-vector ξ_α, it follows that the field Equations (6) are symmetric hyperbolic. This result is guaranteed also if f is substituted by its Taylor expansion up to whatever fixed order greater than 2 but with the Lagrange multipliers as independent variables. Moreover, as usual in Rational Extended Thermodynamics, the closure is determined except for the 4-potentials h'^{α}.

Now people doesn't like these variables and requires that the cloure is expressed in terms of variables with a clear physical meaning. In reality this is not reasonable: It is like saying, in the geometric framework that the parametric equation of a curve or a surface are not significant. Another objection is made that, not knowing these variables, we cannot know their boundary values necessary to solve the field equations. This objection is also unfounded because, knowing the law that binds the physical variables to the Lagrange multipliers, from the boundary conditions for the phyisical variables we can deduce those for the Lagrange multipliers and then solve the field equations. However, in order to meet the commonly accepted taste, we will make the change of variables in the next subsections, from the Lagrange multipliers to the phyisical variables.

Obviouly, hyperbolicity is not compromised by an invertible change of independent variables. Unfortunately, up to now nobody was able to do this exactly and we too will be content to do it in an approximate way, at first order with respect to equilibrium. Due to this approximation, the hyperbolicity requirement will be satisfied only in a neighborhood of equilibrium called "the hyperbolicity zone" (See [13-16]). But this cannot be adduced as proof of the weakness of the model; it is only a proof of our mathematical inability to perform this transformation without introducing approximations. We certainly cannot expect Nature to bow to our mathematical weakness.

5.1. The Variables at Equilibrium

Equilibrium is defined as the state governed only by $(6)_1$ with $r = 0,1$, i.e., the conservation laws of mass and of momentum-energy (Euler Equations), i.e., the subsystem of (6) with $S = -1$ in the sense of Boillat and Ruggeri (1997) [8]. It follows that $\lambda^E_{\alpha_1 \cdots \alpha_r} = 0$, $\mu^E_{\alpha_1 \cdots \alpha_s} = 0$ for $r = 2, \cdots S+2$, $s = 0, \cdots S$. Moreover, from the Representation Theorems we have

$$A^{\alpha} = m\, n\, U^{\alpha} \,, A^{\alpha\beta} = \frac{e}{c^2}\, U^{\alpha} U^{\beta} + p\, h^{\alpha\beta} \,, U^{\alpha} U_{\alpha} = c^2 \,, h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2}\, U^{\alpha} U^{\beta} \,,$$

whose phyisical meaning is obvious: n is the number density, p is the pressure and e the energy.

From $(6)_1$ with $r = 0$ it follows that λ^E_{α} is parallel to U_{α}; so, by calling $\frac{1}{T}$ the coefficient, we have that;

$$\lambda^E_{\alpha} = \frac{U_{\alpha}}{T} \,.$$

The physical meaning of T is evident; it is the absolute temperature.

Of $(6)_1$ with $r = 0,1$ there remain;

$$m\, n\, U^{\alpha} = m\, c\, e^{-1-\frac{m}{k_B}\lambda^E} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{p^{\mu} U_{\mu}}{m\, c^2}\left(1+\frac{\mathcal{I}}{m\, c^2}\right)}\, p^{\alpha} \phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\, \mathcal{I}^R\, d\, \mathcal{I}^V\, d\, \vec{P}$$

$$\frac{e}{c^2}\, U^{\alpha} U^{\beta} + p\, h^{\alpha\beta} = c\, e^{-1-\frac{m}{k_B}\lambda^E} \int_{\Re^3} \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{p^{\mu} U_{\mu}}{m\, c^2}\left(1+\frac{\mathcal{I}}{m\, c^2}\right)}\, p^{\alpha} p^{\beta} \left(1 + \frac{\mathcal{I}}{m\, c^2}\right) \cdot$$
$$\cdot\, \phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)\, d\, \mathcal{I}^R\, d\, \mathcal{I}^V\, d\, \vec{P} \,.$$

The first one of these equations must be imposed only through its contraction with U_{α} and the second one through its contractions with $U_{\alpha} U_{\beta}$ and $h_{\alpha\beta}$. So we obtain;

$$n = 4\, \pi m^3\, e^{-1-\frac{m}{k_B}\lambda^E}\, \overline{J^*_{2,1}} \quad , \quad p = \frac{n\, m\, c^2}{\gamma} = n\, k_B\, T \,,$$
$$e = n\, m\, c^2\, \frac{\overline{J^*_{2,2}\left(1+\frac{\mathcal{I}}{m\, c^2}\right)}}{\overline{J^*_{2,1}}} \,, \tag{22}$$

Where the integration in $d\, \vec{P}$ has been performed and overlined terms denote that they are multiplied by $\phi(\mathcal{I}^R)\, \psi(\mathcal{I}^V)$ and then integrated in $d\, \mathcal{I}^R\, d\, \mathcal{I}^V$. Moreover, in $(22)_{2,3}$ the term λ^E has been eliminated by use of $(22)_1$, $\gamma = \frac{m\, c^2}{k_B T}$.

$$J_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma \cosh s} \sinh^m s\, \cosh^n s\, d\, s$$
$$\gamma^* = \gamma \left(1 + \frac{\mathcal{I}}{m\, c^2}\right) \quad , \quad J^*_{m,n} = J_{m,n}(\gamma^*) \,.$$

The Equation $(22)_3$ is the generalization of the Synge energy to the case of polyatomic gases with rotational and vibrational modes; in the case with only one mode it is the same of Equation (42) of Pennisi and Ruggeri (2017) [11]. The other functions in Equation (7) don't play a role at equilibrium but nothing prevent us from calculating them and they will be useful in the sequel. They are:

$$A_E^{\alpha_1\cdots\alpha_{r+1}} = \sum_{q=0}^{\left[\frac{r+1}{2}\right]} a_{q,r}(\gamma) h^{(\alpha_1\alpha_2} \cdots h^{\alpha_{2q-1}\alpha_{2q}} U^{\alpha_{2q+1}} \cdots U^{\alpha_{r+1})} ,$$
$$A_{VE}^{\alpha_1\cdots\alpha_{s+1}} = \sum_{q=0}^{\left[\frac{s+1}{2}\right]} a_{q,s}^V(\gamma) h^{(\alpha_1\alpha_2} \cdots h^{\alpha_{2q-1}\alpha_{2q}} U^{\alpha_{2q+1}} \cdots U^{\alpha_{s+1})} , \tag{23}$$

Where;

$$a_{q,r} = \binom{r+1}{2q} \frac{n\,m}{2q+1} \frac{c^{2q}}{\overline{J_{2,1}^*}} \overline{J_{2q+2\,,r+1-2q}^*\left(1+\frac{r\,\mathcal{I}}{m\,c^2}\right)} ,$$

$$a_{q,s}^V = \binom{s+1}{2q} \frac{n\,m}{2q+1} \frac{c^{2q}}{\overline{J_{2,1}^*}} \overline{J_{2q+2\,,s+1-2q}^*\left(\frac{2\,\mathcal{I}^V}{m\,c^2}\right)} .$$

These expressions have been found by using the techniques exposed in Carrisi and Pennisi (2013) [17]. We note that $(23)_1$ for $r=0{,}1$ gives the above reported $A_E^{\alpha_1} = V^{\alpha_1}$, $A_E^{\alpha_1\alpha_2} = T_E^{\alpha_1\alpha_2}$ because::

$$a_{0,0} = n\,m\,, \quad a_{0,1} = \frac{e}{c^2}\,, \quad a_{1,1} = \frac{n\,m\,c^2}{3} \frac{\overline{J_{4,0}^*\left(1+\frac{\mathcal{I}}{m\,c^2}\right)}}{\overline{J_{2,1}^*}} = p\,,$$

Where for the last one we have used the property $\gamma J_{4,0}(\gamma) = 3\,J_{2,1}(\gamma)$ from which it follows $\gamma\left(1+\frac{\mathcal{I}}{m\,c^2}\right)J_{4,0}^* = 3\,J_{2,1}^*$. Moreover, $(23)_1$ for $r=2$ and $(23)_2$ for $s=0$ give:

$$A_E^{\alpha_1\alpha_2\alpha_3} = a_{0,2} U^{\alpha_1}U^{\alpha_2}U^{\alpha_3} + a_{1,2} h^{(\alpha_1\alpha_2}U^{\alpha_3} \cdots U^{\alpha_3)}\,, A_{VE}^{\alpha_1} = a_{0,0}^V(\gamma) U^{\alpha_1} .$$

These expressions are the same found in Equation (13) of Carrisi and Pennisi (2019) [9] and the first one of these, in the particular case with only one mode, is the same of Equation (48) in Pennisi and Ruggeri (2017) [11] because;

$$a_{0,2} = A_1^0\,, \quad a_{1,2} = 3\,A_{11}^0\,, \quad a_{0,0}^V = n\,m\, \frac{\overline{J_{2,1}^*\left(1+\frac{2\,\mathcal{I}^V}{m\,c^2}\right)}}{\overline{J_{2,1}^*}} = \frac{c^3}{m\,n} H_V = \frac{\gamma}{m\,n\,c} B_{10} .$$

5.2. The Linear Deviation from Equilibrium

At a first step we will consider as first order deviations from equilibrium the variables π (dynamic pressure), q^α (heat flux), $t^{<\alpha\beta>_3}$ (viscous deviatoric stress), $\lambda_{\alpha_1\cdots\alpha_r}$ for $r=2\,,\cdots\,,S+2$, and $\mu_{\alpha_1\cdots\alpha_s}$ for $s=0\,,\cdots\,,S$. These variables are constrained by $U_\alpha q^\alpha = 0$, $U_\alpha t^{<\alpha\beta>_3} = 0$, $g_{\alpha\beta} t^{<\alpha\beta>_3} = 0$. Also the remaining Lagrange multipliers can be found in terms of a corresponding set of components of $A^{\alpha\alpha_1\cdots\alpha_r}$ and $A_V^{\alpha\alpha_1\cdots\alpha_s}$ (for example, those components whose non relativistic limit gives the variables that, in the classical model, are derivated with respect to time; or, more precisely, their deviations from equilibrium); but this further change can be done in a second step, if it will be considered necessary. Since this amounts only in some complicated systems, we will refrain to report them here.

The change of variables is performed in the following way:

We will use;

$$V^\alpha - V_E^\alpha = 0 \quad , \quad T^{\alpha\beta} - T_E^{\alpha\beta} = \pi\,h^{\alpha\beta} + \frac{2}{c^2}\,q^{(\alpha}U^{\beta)} \;+\; t^{<\alpha\beta>_3} \tag{24}$$

to determine $\lambda - \lambda^E$, $\lambda_\beta - \lambda_\beta^E$, $\lambda_{<\beta\gamma>}$ in terms of n, γ, U^α, π, q^α, $t^{<\alpha\beta>_3}$, $\mu = \frac{1}{4}\,g^{\alpha\beta}\lambda_{\alpha\beta}$, $\lambda_{\alpha_1\cdots\alpha_r}$ for $r=3\,,\cdots\,,S+2$ and $\mu_{\alpha_1\cdots\alpha_s}$ for $s=0\,,\cdots\,,S$. After that, we will substitute these values in $A^{\alpha\alpha_1\cdots\alpha_r} - A_E^{\alpha\alpha_1\cdots\alpha_r}$ and $A_V^{\alpha\alpha_1\cdots\alpha_s} - A_{EV}^{\alpha\alpha_1\cdots\alpha_s}$ so determining the closure, as a consequence, the mass and energy-momentum conservation laws $(6)_1$ with $r=0{,}1$ will be the usual equations with;

$$V^\alpha = m\,n\,U^\alpha \quad , \quad A^{\alpha\beta} = \frac{e}{c^2}\,U^\alpha U^\beta + (p+\pi)\,h^{\alpha\beta} + \frac{2}{c^2}\,q^{(\alpha}U^{\beta)} \;+\; t^{<\alpha\beta>_3} .$$

Now Equations (24) become;

$$\begin{aligned} &A_E^\alpha\,(\lambda-\lambda^E) + A_E^{\alpha\nu}\,(\lambda_\nu - \lambda_\nu^E) + A_E^{\alpha\gamma\delta}\,\lambda_{<\gamma\delta>} + \left(A_E^{\alpha\gamma\delta}\,g_{\gamma\delta}\right)\mu + \\ &+ \sum_{r'=3}^{S+2} A_E^{\alpha\beta_1\cdots\beta_{r'}}\,\lambda_{\beta_1\cdots\beta_{r'}} + \sum_{s'=0}^{S} A_{VE}^{\alpha\beta_1\cdots\beta_{s'}}\,\mu_{\beta_1\cdots\beta_{s'}} = 0\,, \\ &A_E^{\alpha\beta}\,(\lambda-\lambda^E) + \frac{A_{11}^{\alpha\beta\nu}}{m}\,(\lambda_\nu - \lambda_\nu^E) + \frac{A_{12}^{\alpha\beta\gamma\delta}}{m}\,\lambda_{<\gamma\delta>} + \left(\frac{A_{12}^{\alpha\beta\gamma\delta}}{m}\,g_{\gamma\delta}\right)\mu + \\ &+ \sum_{r'=3}^{S+2} \frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\,\lambda_{\beta_1\cdots\beta_{r'}} + \sum_{s'=0}^{S} \frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\,\mu_{\beta_1\cdots\beta_{s'}} = \\ &= -\frac{k_B}{m}\left(\pi\,h^{\alpha\beta} + \frac{2}{c^2}\,q^{(\alpha}U^{\beta)} \;+\; t^{<\alpha\beta>_3}\right) . \end{aligned} \tag{25}$$

Similarly, Equations (7) give;

$$
\begin{aligned}
& A_E^{\alpha\alpha_1\cdots\alpha_r}(\lambda-\lambda^E)+\frac{1}{m}A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}(\lambda_\nu-\lambda_\nu^E)+\frac{1}{m}A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\lambda_{<\gamma\delta>}+\frac{1}{m}\left(A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}g_{\gamma\delta}\right)\mu+ \\
& +\frac{1}{m}\sum_{r'=3}^{S+2}A_{rr'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}}\lambda_{\beta_1\cdots\beta_{r'}}+\frac{1}{m}\sum_{s'=0}^{S}B_{rs'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{s'}}\mu_{\beta_1\cdots\beta_{s'}}=-\frac{k_B}{m}\left(A^{\alpha\alpha_1\cdots\alpha_r}-A_E^{\alpha\alpha_1\cdots\alpha_r}\right), \\
& A_{VE}^{\alpha\alpha_1\cdots\alpha_s}(\lambda-\lambda^E)+\frac{1}{m}B_{s1}^{\alpha\alpha_1\cdots\alpha_s\nu}(\lambda_\nu-\lambda_\nu^E)+\frac{1}{m}B_{s2}^{\alpha\alpha_1\cdots\alpha_s\gamma\delta}\lambda_{<\gamma\delta>}+\frac{1}{m}\left(B_{s2}^{\alpha\alpha_1\cdots\alpha_s\gamma\delta}g_{\gamma\delta}\right)\mu+ \\
& +\frac{1}{m}\sum_{r'=3}^{S+2}B_{sr'}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{r'}}\lambda_{\beta_1\cdots\beta_{r'}}+\frac{1}{m}\sum_{s'=0}^{S}C_{ss'}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{s'}}\mu_{\beta_1\cdots\beta_{s'}}=-\frac{k_B}{m}\left(A_V^{\alpha\alpha_1\cdots\alpha_s}-A_{VE}^{\alpha\alpha_1\cdots\alpha_s}\right).
\end{aligned}
\tag{26}
$$

In Equations (25) and (26) the new tensors appear:

$$
\begin{aligned}
& A_{rr'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}}=\frac{c}{m^{r+r'-2}}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty}f_E\,p^\alpha p^{\alpha_1}\cdots p^{\alpha_r}p^{\beta_1}\cdots p^{\beta_{r'}}\,\cdot \\
& \cdot\left(1+\frac{r\,\mathcal{I}}{m\,c^2}\right)\left(1+\frac{r'\,\mathcal{I}}{m\,c^2}\right)\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\,\mathcal{I}^R\,d\,\mathcal{I}^V\,d\,\vec{P}, \\
& B_{rs}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_s}=\frac{c}{m^{r+s-2}}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty}f_E\,p^\alpha p^{\alpha_1}\cdots p^{\alpha_r}p^{\beta_1}\cdots p^{\beta_s}\left(1+\frac{r\,\mathcal{I}}{m\,c^2}\right)\frac{2\,\mathcal{I}^V}{m\,c^2}\,\cdot \\
& \cdot\,\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\,\mathcal{I}^R\,d\,\mathcal{I}^V\,d\,\vec{P}, \\
& C_{ss'}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{s'}}=\frac{c}{m^{s+s'-2}}\int_{\Re^3}\int_0^{+\infty}\int_0^{+\infty}f_E\,p^\alpha p^{\alpha_1}\cdots p^{\alpha_s}p^{\beta_1}\cdots p^{\beta_{s'}}\left(\frac{2\,\mathcal{I}^V}{m\,c^2}\right)^2\,\cdot \\
& \cdot\,\phi(\mathcal{I}^R)\,\psi(\mathcal{I}^V)\,d\,\mathcal{I}^R\,d\,\mathcal{I}^V\,d\,\vec{P},
\end{aligned}
\tag{27}
$$

Their expressions can be found with the procedure used above or, simply by comparing the definitons of $A_{rr'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}}$ and $A_E^{\alpha\alpha_1\cdots\alpha_r}$ and noting that the former can somehow be obtained from the latter by replacing r with $r+r'$ and multiplying by $m\left(1+\frac{r'\,\mathcal{I}}{m\,c^2}\right)$. So we obtain the first one of the following relations, with coefficients given by Equation $(29)_1$:

$$
\begin{aligned}
& A_{rr'}^{\alpha_1\cdots\alpha_{r+r'+1}}=\sum_{q=0}^{\left[\frac{r+r'+1}{2}\right]}a_{q,r,r'}(\gamma)h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{2q-1}\alpha_{2q}}U^{\alpha_{2q+1}}\cdots U^{\alpha_{r+r'+1})}, \\
& B_{rs'}^{\alpha_1\cdots\alpha_{r+s'+1}}=\sum_{q=0}^{\left[\frac{r+s'+1}{2}\right]}b_{q,r,s'}(\gamma)h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{2q-1}\alpha_{2q}}U^{\alpha_{2q+1}}\cdots U^{\alpha_{r+s'+1})} \\
& C_{ss'}^{\alpha_1\cdots\alpha_{s+s'+1}}=\sum_{q=0}^{\left[\frac{s+s'+1}{2}\right]}c_{q,s,s'}(\gamma)h^{(\alpha_1\alpha_2}\cdots h^{\alpha_{2q-1}\alpha_{2q}}U^{\alpha_{2q+1}}\cdots U^{\alpha_{s+s'+1})}.
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
& a_{q,r,r'}=\binom{r+r'+1}{2q}\frac{m^2n}{2q+1}\frac{c^{2q}}{\overline{\overline{J_{2,1}^*}}}\;\overline{J_{2q+2,r+r'+1-2q}^*\left(1+\frac{r\,\mathcal{I}}{m\,c^2}\right)\left(1+\frac{r'\,\mathcal{I}}{m\,c^2}\right)}, \\
& b_{q,r,s'}=\binom{r+s'+1}{2q}\frac{m^2n}{2q+1}\frac{c^{2q}}{\overline{\overline{J_{2,1}^*}}}\;\overline{J_{2q+2,r+s'+1-2q}^*\left(1+\frac{r\,\mathcal{I}}{m\,c^2}\right)\frac{2\,\mathcal{I}^V}{m\,c^2}}, \\
& c_{q,s,s'}=\binom{s+s'+1}{2q}\frac{m^2n}{2q+1}\frac{c^{2q}}{\overline{\overline{J_{2,1}^*}}}\;\overline{J_{2q+2,s+s'+1-2q}^*\left(\frac{2\,\mathcal{I}^V}{m\,c^2}\right)^2}.
\end{aligned}
\tag{29}
$$

Similarly, by comparing the definitons of $B_{rs}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_s}$ and $A_E^{\alpha\alpha_1\cdots\alpha_r}$, we note that the former can somehow be obtained from the latter by replacing r with $r+s$ and multiplying by $\frac{2\,\mathcal{I}^V}{m\,c^2}$. So we obtain $(28)_2$ with coefficients given by Equation $(29)_2$.

Finaly, by comparing the definitons of $C_{ss'}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{s'}}$ and $A_E^{\alpha\alpha_1\cdots\alpha_s}$, we note that the former can somehow be obtained from the latter by replacing s with $s+s'$ and multiplying by $\frac{1}{m}\frac{2\,\mathcal{I}^V}{m\,c^2}$. So we obtain $(28)_3$ with coefficients given by $(29)_3$. We note that $(27)_1$ with $r=1,\ r'=1$ gives the expression $(15)_1$ of Carrisi and Pennisi (2019) [9] multiplied by m^2; $(27)_1$ with $r=1,\ r'=2$ gives the expression $(15)_2$ of Carrisi and Pennisi (2019) [9] multiplied by m^2; $(27)_1$ with $r=2,\ r'=2$ gives the expression $(15)_3$ of Carrisi and Pennisi (2019) [9] multiplied by m^2. Similarly, $(27)_2$ with $r=1,\ s=0$ gives $B_{10}^{\alpha\alpha_1}=m\,c\,T_V^{\alpha\alpha_1}$, where $T_V^{\alpha\alpha_1}$ is the expression $(15)_4$ of Carrisi and Pennisi (2019) [9]; $(27)_2$ with $r=2,\ s=0$ gives $B_{20}^{\alpha\alpha_1\alpha_2}=m\,c\,A_V^{\alpha\alpha_1\alpha_2}$, where $A_V^{\alpha\alpha_1\alpha_2}$ is the expression $(15)_5$ of Carrisi and Pennisi (2019) [9]. Finally, $(27)_3$ with $s=0,\ s'=0$ gives $C_{00}^{\alpha}=m\,c\,V_{VV}^{\alpha}$ with V_{VV}^{α} defined in $(15)_6$ of Carrisi and Pennisi (2019) [9].

By comparing the correspondent decompositions (28) with (16) of Carrisi and Pennisi (2019) [9], we see that we must have:

$$a_{0,1,1}=m^2B_5\,; \quad a_{1,1,1}=m^2B_4\,; \quad a_{0,1,2}=m^2B_3\,;$$
$$a_{1,1,2}=2\,m^2B_2\,; \quad a_{2,1,2}=\frac{1}{5}m^2B_1\,; \quad a_{0,2,2}=m^2B_8\,;$$
$$a_{1,2,2}=\frac{10}{3}\,m^2B_7\,; \quad a_{2,2,2}=m^2B_6\,; \quad b_{0,1,0}=\frac{m}{c}B_9\,;$$
$$b_{1,1,0}=m\,c\,B_{10}\,; \quad b_{0,2,0}=m\,c\,A_{1V}^0\,; \quad b_{1,2,0}=3\,m\,c\,A_{11V}^0\,; c_{0,0,0}=m\,c\,B_{11}\,,$$

with B_1-B_{11}, A_{1V}^0, A_{11V}^0 given by (17) of Carrisi and Pennisi (2019) [9]. This is confirmed by the present Equation (29). We are now ready to determine $\lambda-\lambda^E$, $\lambda_\beta-\lambda_\beta^E$, $\lambda_{<\beta\gamma>}$ from (26) in terms of $n.\ \gamma$, U^α, π, q^α, $t^{<\alpha\beta>_3}$, $\mu=\frac{1}{4}\,g^{\alpha\beta}\lambda_{\alpha\beta}$, $\lambda_{\alpha_1\cdots\alpha_r}$ for $r=3\,,\cdots\,,S+2$ and $\mu_{\alpha_1\cdots\alpha_s}$ for $s=0\,,\cdots\,,S$. After that, we will substitute them in (27) and obtain the requested closure. To this end, let us contract Equation $(25)_1$ with U_α and Equation $(25)_2$ a first time with $U_\alpha U_\beta$ and a second time with $h_{\alpha\beta}$; so we obtain

$$\begin{aligned}
&nc^2(\lambda-\lambda_E)+\frac{e}{m}\,U^\mu\left(\lambda_\mu-\frac{U_\mu}{T}\right)+\frac{1}{m}\,(A_1^0c^2+A_{11}^0)\,U^\mu U^\nu\lambda_{<\mu\nu<}=\\
&=-\frac{c^2}{m}\,(A_1^0c^2-3A_{11}^0)\,\mu-\textstyle\sum_{r\prime=3}^{S+2}U_\alpha\frac{A_E^{\alpha\beta_1\cdots\beta_{r\prime}}}{m}\,\lambda_{\beta_1\cdots\beta_{r\prime}}-\sum_{s\prime=0}^{S}U_\alpha\frac{A_{VE}^{\alpha\beta_1\cdots\beta_{s\prime}}}{m}\,\mu_{\beta_1\cdots\beta_{s\prime}}\,,\\
&\frac{e}{m}\,c^2(\lambda-\lambda_E)+c^4B_5\,U^\mu\left(\lambda_\mu-\frac{U_\mu}{T}\right)+\left(\frac{1}{3}B_2c^2+B_3c^4\right)U^\mu U^\nu\lambda_{<\mu\nu>}=\\
&=(B_2-B_3c^2)c^4\mu-\textstyle\sum_{r\prime=3}^{S+2}U_\alpha U_\beta\frac{A_{1r\prime}^{\alpha\beta\beta_1\cdots\beta_{r\prime}}}{m}\,\lambda_{\beta_1\cdots\beta_{r\prime}}-\sum_{s\prime=0}^{S}U_\alpha U_\beta\frac{B_{1s\prime}^{\alpha\beta\beta_1\cdots\beta_{s\prime}}}{m}\,\mu_{\beta_1\cdots\beta_{s\prime}}\,,\\
&\frac{p}{m}\,(\lambda-\lambda_E)+\frac{1}{3}B_4U^\mu\left(\lambda_\mu-\frac{U_\mu}{T}\right)+\left(\frac{1}{3}B_2+\frac{1}{9}\frac{B_1}{c^2}\right)U^\mu U^\nu\lambda_{<\mu\nu>}=\\
&=-\frac{k_B}{m^2}\,\pi+\frac{1}{3}\,(B_1-B_2c^2)\,\mu-\frac{1}{3}\sum_{r\prime=3}^{S+2}h_{\alpha\beta}\frac{A_{1r\prime}^{\alpha\beta\beta_1\cdots\beta_{r\prime}}}{m}\,\lambda_{\beta_1\cdots\beta_{r\prime}}-\frac{1}{3}\sum_{s\prime=0}^{S}h_{\alpha\beta}\frac{B_{1s\prime}^{\alpha\beta\beta_1\cdots\beta_{s\prime}}}{m}\,\mu_{\beta_1\cdots\beta_{s\prime}}\,.
\end{aligned} \tag{30}$$

If we calculate this system in $\mu=0$, $\lambda_{\beta_1\cdots\beta_{r\prime}}=0$, $\mu_{\beta_1\cdots\beta_{s\prime}}=0$ we obtain exactly the system $(A.10)_{1-3}$ of Pennisi and Ruggeri (2017) [11]. Obviously, the matrix of coefficients is the same of that reported in $(A.11)_1$ of Pennisi and Ruggeri (2017) [11], i.e.,

$$\widetilde{D}^\pi=\begin{pmatrix} nc^2 & \frac{e}{m} & \frac{1}{m}\,(A_1^0c^2+A_{11}^0)\\ \frac{e}{m}\,c^2 & c^4\,B_5 & \frac{1}{3}B_2c^2+B_3c^4\\ \frac{p}{m} & \frac{1}{3}B_4 & \frac{1}{3}B_2+\frac{1}{9}\frac{B_1}{c^2}\end{pmatrix}. \tag{31}$$

So, we can define $\widetilde{D}_{ij}^\pi$ the algebraic complement of its element in the line i, coulumn j and, by using the Kramer' s theorem, we find;

$$\begin{aligned}
&\lambda-\lambda_E=\frac{\widetilde{D}_{31}^\pi}{|\widetilde{D}^\pi|}\Bigg(-\frac{k_B}{m^2}\,\pi+\frac{1}{3}\,(B_1-B_2c^2)\,\mu-\\
&\frac{1}{3}\sum_{r\prime=3}^{S+2}h_{\alpha\beta}\frac{A_{1r\prime}^{\alpha\beta\beta_1\cdots\beta_{r\prime}}}{m}\,\lambda_{\beta_1\cdots\beta_{r\prime}}-\frac{1}{3}\sum_{s\prime=0}^{S}h_{\alpha\beta}\frac{B_{1s\prime}^{\alpha\beta\beta_1\cdots\beta_{s\prime}}}{m}\,\mu_{\beta_1\cdots\beta_{s\prime}}\Bigg)+\\
&+\frac{\widetilde{D}_{21}^\pi}{|\widetilde{D}^\pi|}\left((B_2-B_3c^2)c^4\mu-\sum_{r\prime=3}^{S+2}U_\alpha U_\beta\frac{A_{1r\prime}^{\alpha\beta\beta_1\cdots\beta_{r\prime}}}{m}\,\lambda_{\beta_1\cdots\beta_{r\prime}}-\sum_{s\prime=0}^{S}U_\alpha U_\beta\frac{B_{1s\prime}^{\alpha\beta\beta_1\cdots\beta_{s\prime}}}{m}\,\mu_{\beta_1\cdots\beta_{s\prime}}\right)+\\
&+\frac{\widetilde{D}_{11}^\pi}{|\widetilde{D}^\pi|}\left(-\frac{c^2}{m}\,(A_1^0c^2-3A_{11}^0)\,\mu-\sum_{r\prime=3}^{S+2}U_\alpha\frac{A_E^{\alpha\beta_1\cdots\beta_{r\prime}}}{m}\,\lambda_{\beta_1\cdots\beta_{r\prime}}-\sum_{s\prime=0}^{S}U_\alpha\frac{A_{VE}^{\alpha\beta_1\cdots\beta_{s\prime}}}{m}\,\mu_{\beta_1\cdots\beta_{s\prime}}\right),
\end{aligned} \tag{32}$$

$$\begin{aligned}
&U^\mu(\lambda_\mu-\lambda_{E\mu})=\frac{\widetilde{D}_{32}^\pi}{|\widetilde{D}^\pi|}\Bigg(-\frac{k_B}{m^2}\,\pi+\frac{1}{3}\,(B_1-B_2c^2)\,\mu-\\
&\frac{1}{3}\sum_{r\prime=3}^{S+2}h_{\alpha\beta}\frac{A_{1r\prime}^{\alpha\beta\beta_1\cdots\beta_{r\prime}}}{m}\,\lambda_{\beta_1\cdots\beta_{r\prime}}-\frac{1}{3}\sum_{s\prime=0}^{S}h_{\alpha\beta}\frac{B_{1s\prime}^{\alpha\beta\beta_1\cdots\beta_{s\prime}}}{m}\,\mu_{\beta_1\cdots\beta_{s\prime}}\Bigg)+
\end{aligned}$$

$$+\frac{\widetilde{D}_{22}^{\pi}}{|\widetilde{D}^{\pi}|}\left((B_2-B_3c^2)c^4\mu-\sum_{r'=3}^{S+2}U_\alpha U_\beta\frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-\sum_{s'=0}^{S}U_\alpha U_\beta\frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\right)+$$

$$+\frac{\widetilde{D}_{21}^{\pi}}{|\widetilde{D}^{\pi}|}\left(-\frac{c^2}{m}(A_1^0c^2-3A_{11}^0)\mu-\sum_{r'=3}^{S+2}U_\alpha\frac{A_E^{\alpha\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-\sum_{s'=0}^{S}U_\alpha\frac{A_{VE}^{\alpha\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\right),$$

$$U^\mu U^\nu\lambda_{<\mu\nu>}=\frac{\widetilde{D}_{33}^{\pi}}{|\widetilde{D}^{\pi}|}\Big(-\frac{k_B}{m^2}\pi+\frac{1}{3}(B_1-B_2c^2)\mu-$$

$$\frac{1}{3}\sum_{r'=3}^{S+2}h_{\alpha\beta}\frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-\frac{1}{3}\sum_{s'=0}^{S}h_{\alpha\beta}\frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\Big)+$$

$$+\frac{\widetilde{D}_{23}^{\pi}}{|\widetilde{D}^{\pi}|}\left((B_2-B_3c^2)c^4\mu-\sum_{r'=3}^{S+2}U_\alpha U_\beta\frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-\sum_{s'=0}^{S}U_\alpha U_\beta\frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\right)+$$

$$+\frac{\widetilde{D}_{13}^{\pi}}{|\widetilde{D}^{\pi}|}\left(-\frac{c^2}{m}(A_1^0c^2-3A_{11}^0)\mu-\sum_{r'=3}^{S+2}U_\alpha A_E^{\alpha\beta_1\cdots\beta_{r'}}\lambda_{\beta_1\cdots\beta_{r'}}-\sum_{s'=0}^{S}U_\alpha A_{VE}^{\alpha\beta_1\cdots\beta_{s'}}\mu_{\beta_1\cdots\beta_{s'}}\right),$$

If we calculate these expressions in $\mu=0$, $\lambda_{\beta_1\cdots\beta_{r'}}=0$, $\mu_{\beta_1\cdots\beta_{s'}}=0$, we obtain exactly those reported in the equations subsequent to (61) of Pennisi and Ruggeri (2017) [11]. We consider now Equation $(25)_1$ contracted by h_α^δ and Equation $(25)_2$ contracted $h_\alpha^\delta\, U_\beta$. So we obtain the system:

$$\begin{pmatrix}\frac{p}{m} & 2\,\frac{A_{11}^0}{m}\\ \frac{1}{3}B_4c^2 & \frac{2}{3}B_2\,c^2\end{pmatrix}\begin{pmatrix}h^{\delta\mu}\left(\lambda_\mu-\frac{U_\mu}{T}\right)\\ h^{\delta\mu}U^\nu\lambda_{<\mu\nu>}\end{pmatrix}=$$

$$=\begin{pmatrix}-h_\alpha^\delta\sum_{r'=3}^{S+2}\frac{A_E^{\alpha\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-h_\alpha^\delta\sum_{s'=0}^{S}\frac{A_{VE}^{\alpha\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\\ -\frac{k_B}{m^2}q^\delta-\sum_{r'=3}^{S+2}h_\alpha^\delta U_\beta\frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-\sum_{s'=0}^{S}h_\alpha^\delta U_\beta\frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}.\end{pmatrix}$$

By calling $\widetilde{D}^q$ the determinant of the coefficients we can use the Kramer' s theorem and find;

$$h^{\delta\mu}\left(\lambda_\mu-\frac{U_\mu}{T}\right)=$$

$$=-\frac{2\,A_{11}^0}{m\,\widetilde{D}^q}\left(-\frac{k_B}{m^2}q^\delta-\sum_{r'=3}^{S+2}h_\alpha^\delta U_\beta\frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-\sum_{s'=0}^{S}h_\alpha^\delta U_\beta\frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\right)+$$

$$+\frac{2}{3\,\widetilde{D}^q}B_2\,c^2\left(-h_\alpha^\delta\sum_{r'=3}^{S+2}\frac{A_E^{\alpha\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-h_\alpha^\delta\sum_{s'=0}^{S}\frac{A_{VE}^{\alpha\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\right),$$

$$h^{\delta\mu}U^\nu\lambda_{<\mu\nu>}=\frac{p}{m\,\widetilde{D}^q}\left(-\frac{k_B}{m^2}q^\delta-\sum_{r'=3}^{S+2}h_\alpha^\delta U_\beta\frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-\sum_{s'=0}^{S}h_\alpha^\delta U_\beta\frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\right),$$

$$+\frac{1}{3\,\widetilde{D}^q}B_4\,c^2\left(-h_\alpha^\delta\sum_{r'=3}^{S+2}\frac{A_E^{\alpha\beta_1\cdots\beta_{r'}}}{m}\lambda_{\beta_1\cdots\beta_{r'}}-h_\alpha^\delta\sum_{s'=0}^{S}\frac{A_{VE}^{\alpha\beta_1\cdots\beta_{s'}}}{m}\mu_{\beta_1\cdots\beta_{s'}}\right).$$

If we calculate these expressions in $\mu=0$, $\lambda_{\beta_1\cdots\beta_{r'}}=0$, $\mu_{\beta_1\cdots\beta_{s'}}=0$, we obtain exactly Equation $(A.14)_{1,2}$ of Pennisi and Ruggeri (2017) [11]. Finally, Equation $(25)_2$ contracted $h_\alpha^{<\delta}\,h_\beta^{\theta>_3}$ gives

$$h_\mu^{<\delta}\,h_\nu^{\theta>_3}\,\lambda^{<\mu\nu>}=$$

$$= \frac{15}{2\,B_1}\left(-\frac{k_B}{m^2}\,t^{<\delta\theta>_3} - \sum_{r'=3}^{S+2} h_\alpha^{<\delta}\,h_\beta^{\theta>_3}\,\frac{A_{1r'}^{\alpha\beta\beta_1\cdots\beta_{r'}}}{m}\,\lambda_{\beta_1\cdots\beta_{r'}} - \sum_{s'=0}^{S} h_\alpha^{<\delta}\,h_\beta^{\theta>_3}\,\frac{B_{1s'}^{\alpha\beta\beta_1\cdots\beta_{s'}}}{m}\,\mu_{\beta_1\cdots\beta_{s'}}\right).$$

If we calculate these expressions in $\mu = 0$, $\lambda_{\beta_1\cdots\beta_{r'}} = 0$, $\mu_{\beta_1\cdots\beta_{s'}} = 0$, we obtain exactly Equation (A.15) $_1$ of Pennisi and Ruggeri (2017) [11]. Now we have to substitute all these results in (26) and, to thi end, we will use the identies:

$$\lambda_\nu - \lambda_\nu^E = -\left(\lambda_\mu - \lambda_\mu^E\right)h_\nu^\mu + \left[\left(\lambda_\mu - \lambda_\mu^E\right)U^\mu\right]\frac{U_\nu}{c^2},$$

$$\lambda_{<\mu\nu>} = \lambda_{<\alpha\beta>}\,h_{<\mu}^{\alpha}\,h_{\nu>_3}^{\beta} - \frac{2}{c^2}\,\lambda_{<\alpha\beta>}\,U^\alpha\,h_{(\mu}^{\beta}\,U_{\nu)} + \frac{\lambda_{<\alpha\beta>}U^\alpha U^\beta}{c^2}\left(U_\mu U_\nu + \frac{1}{3}\,h_{\mu\nu}\right).$$

So we obtain:

$$A^{\alpha\alpha_1\cdots\alpha_r} - A_E^{\alpha\alpha_1\cdots\alpha_r} = \mathcal{A}_\pi^{\alpha\alpha_1\cdots\alpha_r}\,\pi + \mathcal{A}_\mu^{\alpha\alpha_1\cdots\alpha_r}\,\mu + \mathcal{A}_q^{\alpha\alpha_1\cdots\alpha_r\delta}\,q_\delta + \mathcal{A}_t^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\,t_{<\gamma\delta>_3} + \tag{33}$$

$$+\sum_{r'=3}^{S+2}\mathcal{A}_\lambda^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}}\,\lambda_{\beta_1\cdots\beta_{r'}} + \sum_{s'=0}^{S}\mathcal{A}_\mu^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{s'}}\,\mu_{\beta_1\cdots\beta_{s'}}\,,$$

$$A_V^{\alpha\alpha_1\cdots\alpha_s} - A_{VE}^{\alpha\alpha_1\cdots\alpha_s} = \mathcal{A}_{V\pi}^{\alpha\alpha_1\cdots\alpha_s}\,\pi + \mathcal{A}_{V\mu}^{\alpha\alpha_1\cdots\alpha_s}\,\mu + \mathcal{A}_{Vq}^{\alpha\alpha_1\cdots\alpha_s\delta}\,q_\delta + \mathcal{A}_{Vt}^{\alpha\alpha_1\cdots\alpha_s\gamma\delta}\,t_{<\gamma\delta>_3} +$$

$$+\sum_{r'=3}^{S+2}\mathcal{A}_{V\lambda}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{r'}}\,\lambda_{\beta_1\cdots\beta_{r'}} + \sum_{s'=0}^{S}\mathcal{A}_{V\mu}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{s'}}\,\mu_{\beta_1\cdots\beta_{s'}}\,,$$

Where;

$$\mathcal{A}_\pi^{\alpha\alpha_1\cdots\alpha_r} = A_E^{\alpha\alpha_1\cdots\alpha_r}\,\frac{\widetilde{D}_{31}^{\pi}}{m\left|\widetilde{D}^{\pi}\right|} + A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}\,\frac{U_\nu}{c^2}\,\frac{\widetilde{D}_{32}^{\pi}}{m^2\left|\widetilde{D}^{\pi}\right|} + A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\left(U_\gamma U_\delta + \frac{1}{3}h_{\gamma\delta}\right)\frac{\widetilde{D}_{33}^{\pi}}{m^2\,c^2\left|\widetilde{D}^{\pi}\right|},$$

$$\mathcal{A}_\mu^{\alpha\alpha_1\cdots\alpha_r} =$$

$$= -\frac{m}{k_B}\,A_E^{\alpha\alpha_1\cdots\alpha_r}\left[\frac{\widetilde{D}_{31}^{\pi}}{3\left|\widetilde{D}^{\pi}\right|}(B_1 - B_2c^2) + \frac{\widetilde{D}_{21}^{\pi}}{\left|\widetilde{D}^{\pi}\right|}(B_2 - B_3c^2)c^4 - \frac{\widetilde{D}_{11}^{\pi}}{m\left|\widetilde{D}^{\pi}\right|}(A_1^0c^2 - 3A_{11}^0)c^2\right] -$$

$$\frac{1}{k_B}\,A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}\,\frac{U_\nu}{c^2}\left[\frac{\widetilde{D}_{32}^{\pi}}{3\left|\widetilde{D}^{\pi}\right|}(B_1 - B_2c^2) + \frac{\widetilde{D}_{22}^{\pi}}{\left|\widetilde{D}^{\pi}\right|}(B_2 - B_3c^2)c^4 - \frac{\widetilde{D}_{21}^{\pi}}{m\left|\widetilde{D}^{\pi}\right|}(A_1^0c^2 - 3A_{11}^0)c^2\right] -$$

$$\frac{1}{k_B}\,A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\left(U_\gamma U_\delta + \frac{1}{3}h_{\gamma\delta}\right)\left[\frac{\widetilde{D}_{33}^{\pi}}{3\,c^2\left|\widetilde{D}^{\pi}\right|}(B_1 - B_2c^2) + \frac{\widetilde{D}_{23}^{\pi}}{\left|\widetilde{D}^{\pi}\right|}(B_2 - B_3c^2)c^2 -\right.$$

$$\left.\frac{\widetilde{D}_{13}^{\pi}}{m\left|\widetilde{D}^{\pi}\right|}(A_1^0c^2 - 3A_{11}^0)\right] - \frac{1}{k_B}\,A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\,g_{\gamma\delta}\,,\qquad \mathcal{A}_q^{\alpha\alpha_1\cdots\alpha_r\delta} = A_{r1}^{\alpha\alpha_1\cdots\alpha_r\delta}\,\frac{2\,A_{11}^0}{m^3\,\widetilde{D}^q} - \frac{2\,p}{m^3c^2\widetilde{D}^q}A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\,U_\gamma\,,$$

$$\mathcal{A}_t^{\alpha\alpha_1\cdots\alpha_r\gamma\delta} = A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\,\frac{15}{2\,m^2\,B_1}\,,\qquad \mathcal{A}_\lambda^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}} = -\frac{1}{k_B}\,A_{rr'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}} +$$

$$+\frac{A_E^{\alpha\alpha_1\cdots\alpha_r}}{m}\left[\frac{\widetilde{D}_{31}^{\pi}}{3\left|\widetilde{D}^{\pi}\right|}\,h_{\mu\nu}\,A_{1r'}^{\mu\nu\beta_1\cdots\beta_{r'}} + \frac{\widetilde{D}_{21}^{\pi}}{\left|\widetilde{D}^{\pi}\right|}\,U_\mu U_\nu\,A_{1r'}^{\mu\nu\beta_1\cdots\beta_{r'}} + \frac{\widetilde{D}_{11}^{\pi}}{\left|\widetilde{D}^{\pi}\right|}\,U_\mu\,A_E^{\mu\beta_1\cdots\beta_{r'}}\right] +$$

$$+\frac{A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}}{m\,k_B}\,\frac{U_\nu}{c^2}\left[\frac{\widetilde{D}_{32}^{\pi}}{3\left|\widetilde{D}^{\pi}\right|}h_{\mu\vartheta}\,A_{1r'}^{\mu\vartheta\beta_1\cdots\beta_{r'}} + \frac{\widetilde{D}_{22}^{\pi}}{\left|\widetilde{D}^{\pi}\right|}\,U_\mu U_\vartheta\,A_{1r'}^{\mu\vartheta\beta_1\cdots\beta_{r'}} + \frac{\widetilde{D}_{21}^{\pi}}{\left|\widetilde{D}^{\pi}\right|}\,U_\mu\,A_E^{\mu\beta_1\cdots\beta_{r'}}\right] +$$

$$+\frac{2}{k_B}\,A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}\left[\frac{A_{11}^0}{m^2\,\widetilde{D}^q}\,h_{\nu\mu}\,U_\vartheta\,A_{1r'}^{\nu\vartheta\beta_1\cdots\beta_{r'}} - \frac{B_2\,c^2}{3\,m\,\widetilde{D}^q}\,h_{\nu\mu}\,A_E^{\mu\beta_1\cdots\beta_{r'}}\right] +$$

$$+\frac{1}{k_B}A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}A_{1r'}^{\vartheta\beta\beta_1\cdots\beta_{r'}}\left[\frac{15}{2\,m\,B_1}h_{\vartheta<\gamma}h_{\delta>_3\beta}-\frac{2\,p}{m\,c^2\widetilde{D}^q}U_\delta\,h_{\gamma\vartheta}\,U_\beta-\right.$$

$$\left.\frac{\widetilde{D}_{33}^\pi}{3\,m\,c^2\left|\widetilde{D}^\pi\right|}\left(U_\gamma U_\delta+\frac{1}{3}h_{\gamma\delta}\right)h_{\vartheta\beta}+\frac{\widetilde{D}_{23}^\pi}{m\,c^2\left|\widetilde{D}^\pi\right|}\left(U_\gamma U_\delta+\frac{1}{3}h_{\gamma\delta}\right)U_\vartheta U_\beta\right]+$$

$$+\frac{1}{k_B}A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}A_E^{\vartheta\beta_1\cdots\beta_{r'}}\left[-\frac{2\,B_4}{3\,m\,\widetilde{D}^q}U_\delta\,h_{\gamma\vartheta}+\frac{\widetilde{D}_{13}^\pi}{c^2\left|\widetilde{D}^\pi\right|}\left(U_\gamma U_\delta+\frac{1}{3}h_{\gamma\delta}\right)U_\vartheta\right],$$

$$\mathcal{A}_\mu^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{s'}}=-\frac{1}{k_B}B_{rs'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{s'}}+$$

$$+\frac{1}{k_B}A_E^{\alpha\alpha_1\cdots\alpha_r}\left[\frac{\widetilde{D}_{31}^\pi}{3\left|\widetilde{D}^\pi\right|}h_{\vartheta\beta}B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}+\frac{\widetilde{D}_{21}^\pi}{\left|\widetilde{D}^\pi\right|}U_\vartheta U_\beta B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}-\frac{\widetilde{D}_{11}^\pi}{\left|\widetilde{D}^\pi\right|}U_\vartheta A_E^{\vartheta\beta_1\cdots\beta_{s'}}\right]+$$

$$+\frac{A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}}{k_B}\frac{U_\nu}{c^2}\left[\frac{\widetilde{D}_{33}^\pi}{3\left|\widetilde{D}^\pi\right|}h_{\vartheta\beta}\frac{B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}}{m}+\frac{\widetilde{D}_{23}^\pi}{\left|\widetilde{D}^\pi\right|}U_\beta U_\vartheta\frac{B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}}{m}+\frac{\widetilde{D}_{13}^\pi}{\left|\widetilde{D}^\pi\right|}U_\vartheta A_{VE}^{\vartheta\beta_1\cdots\beta_{s'}}\right]+$$

$$+\frac{2}{k_B}\frac{A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}}{m}\left[\frac{A_{11}^0}{m\,\widetilde{D}^q}h_{\nu\vartheta}U_\beta B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}-\frac{B_2\,c^2}{3\,\widetilde{D}^q}h_{\nu\vartheta}A_{VE}^{\vartheta\beta_1\cdots\beta_{s'}}\right]+$$

$$+\frac{1}{k_B}\frac{15}{2\,m\,B_1}A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}h_{\gamma<\vartheta}h_{\beta>_3\delta}+$$

$$+\frac{1}{k_B\,c^2}A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\left(U_\gamma U_\delta+\frac{1}{3}h_{\gamma\delta}\right)\left[\frac{\widetilde{D}_{33}^\pi}{3\,m\left|\widetilde{D}^\pi\right|}h_{\vartheta\beta}B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}+\frac{\widetilde{D}_{23}^\pi}{m\left|\widetilde{D}^\pi\right|}B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}U_\vartheta U_\beta+\right.$$

$$\left.+\frac{\widetilde{D}_{13}^\pi}{\left|\widetilde{D}^\pi\right|}U_\vartheta A_{VE}^{\vartheta\beta_1\cdots\beta_{s'}}\right]+\frac{4}{k_B\,c^2}A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}U_\gamma\left[\frac{A_{11}^0}{m^2\,\widetilde{D}^q}U_\beta\,h_{\delta\vartheta}B_{1s'}^{\vartheta\beta\beta_1\cdots\beta_{s'}}-\frac{B_2\,c^2}{3\,m\,\widetilde{D}^q}h_{\delta\vartheta}A_{VE}^{\vartheta\beta_1\cdots\beta_{s'}}\right].$$

The expressions of $\mathcal{A}_{V\pi}^{\alpha\alpha_1\cdots\alpha_s}$, $\mathcal{A}_{V\mu}^{\alpha\alpha_1\cdots\alpha_s}$, $\mathcal{A}_{Vq}^{\alpha\alpha_1\cdots\alpha_s\delta}$, $\mathcal{A}_{Vt}^{\alpha\alpha_1\cdots\alpha_s\gamma\delta}$, $\mathcal{A}_{V\lambda}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{r'}}$ $\mathcal{A}_{V\mu}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{s'}}$ can be obtained from the above ones by substituting $A_E^{\alpha\alpha_1\cdots\alpha_r}$ with $A_{VE}^{\alpha\alpha_1\cdots\alpha_s}$, $A_{r1}^{\alpha\alpha_1\cdots\alpha_r\nu}$ with $B_{s1}^{\alpha\alpha_1\cdots\alpha_s\nu}$, $A_{r2}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}$ with $B_{s2}^{\alpha\alpha_1\cdots\alpha_s\gamma\delta}$, $A_{rr'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}}$ with $B_{sr'}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{r'}}$, $B_{rs'}^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{s'}}$ with $C_{ss'}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{s'}}$.

In fact, this is what comes out from the comparison between $(26)_1$ and $(26)_2$; obviously, the contribute of the Lagrange multipliers is the same so that nothing else must be changed. So we avoid to report such expressions for the sake of brevity.

The Equations (33) jointly with (23) give the requested closure. Obviously, in Equations (33) the Lagrange multipliers μ, $\lambda_{\beta_1\cdots\beta_{r'}}$, $\mu_{\beta_1\cdots\beta_{s'}}$ still appear between the independent variables. If we want to express them too in terms of physical variables, we have firstly to clarify what these physical variables are besides those already introduced. In my opinion they are those whose non relativistic limit gives the variables which are derivated respect to time, or still better, the deviations from their equilibrium value. In other words, we have to consider the equations:

$$\Delta=\frac{1}{c^4}U_\alpha U_{\alpha_1}U_{\alpha_2}\left(A^{\alpha\alpha_1\alpha_2}-A_E^{\alpha\alpha_1\alpha_2}\right), \tag{34}$$

$$\Delta^{\alpha_1\cdots\alpha_r}=U_\alpha\left(A^{\alpha\alpha_1\cdots\alpha_r}-A_E^{\alpha\alpha_1\cdots\alpha_r}\right)=U_\alpha\,\mathcal{A}_\pi^{\alpha\alpha_1\cdots\alpha_r}\pi+U_\alpha\,\mathcal{A}_\mu^{\alpha\alpha_1\cdots\alpha_r}\mu+U_\alpha\,\mathcal{A}_q^{\alpha\alpha_1\cdots\alpha_r\delta}q_\delta+ \tag{35}$$

$$+U_\alpha\,\mathcal{A}_t^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\,t_{<\gamma\delta>_3}+\sum_{r'=3}^{S+2}U_\alpha\,\mathcal{A}_\lambda^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{r'}}\lambda_{\beta_1\cdots\beta_{r'}}+\sum_{s'=0}^{S}U_\alpha\,\mathcal{A}_\mu^{\alpha\alpha_1\cdots\alpha_r\beta_1\cdots\beta_{s'}}\mu_{\beta_1\cdots\beta_{s'}},$$

$$\Delta_V^{\alpha_1\cdots\alpha_s}=U_\alpha\left(A_V^{\alpha\alpha_1\cdots\alpha_s}-A_{VE}^{\alpha\alpha_1\cdots\alpha_s}\right)=U_\alpha\,\mathcal{A}_{V\pi}^{\alpha\alpha_1\cdots\alpha_s}\pi+U_\alpha\,\mathcal{A}_{V\mu}^{\alpha\alpha_1\cdots\alpha_s}\mu+U_\alpha\,\mathcal{A}_{Vq}^{\alpha\alpha_1\cdots\alpha_s\delta}q_\delta+$$

$$+U_\alpha\,\mathcal{A}_{Vt}^{\alpha\alpha_1\cdots\alpha_r\gamma\delta}\,t_{<\gamma\delta>_3}+\sum_{r'=3}^{S+2}U_\alpha\,\mathcal{A}_{V\lambda}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{r'}}\lambda_{\beta_1\cdots\beta_{r'}}+\sum_{s'=0}^{S}U_\alpha\,\mathcal{A}_{V\mu}^{\alpha\alpha_1\cdots\alpha_s\beta_1\cdots\beta_{s'}}\mu_{\beta_1\cdots\beta_{s'}}.$$

The left hand sides of these equations are the additional physical variables; Equations (34) for $r = 3, \cdots S+2$ and $s = 0, \cdots S$ have to be used to determine μ, $\lambda_{\beta_1 \cdots \beta_r}$, $\mu_{\beta_1 \cdots \beta_s}$, in terms of the physical variables. The result has to be substituted in (33) so obtaining the closure all in terms of physical variables. Can we do this? Yes, we can. But the equations present in this article are complicated enough to want to burden them further. Therefore we refrain from doing it. In any case, when we want to make a practical application of the model, we must first choose in harmony with the experimental results the number S to stop at. In this case, since S is a given number, these further steps can be carried out easily. So the last step in the first part of the flowchart present in the Introduction has been obtained, i.e., the closure of the present general relativistic model (6).

6. Conclusions

In this article, it was found that the relativistic counterpart of the classical model for polyatomic gases takes into account both the vibrational and rotational modes. As is common in Extended Thermodynamics, in the balance equations not only independent variables appear but also other additional tensors; the closure is obtained when the expressions of these tensors are found as functions of the independent variables. This end is here reached by imposing universal principles such as the Entropy Principle, the Maximum Entropy Principle, and, obviously, the covariance of all the equations and the variables involved. As a bonus, the field equations assume the symmetric form and are hyperbolic; this is important because assures the respect of the cause and effect principle and the fact that the wave velocities don't exceed the speed of light. Another nice mathematical property in this way is the continuous dependence on the initial data.

The validity of the present model has already been tested because in the simplest case of 16 moments, it coincides with that already known in the literature. Certainly, the field equations have become somewhat complicated due to the fact that independent variables more appealing to the common reader have been chosen. If we use the Lagrange multipliers as independent variables, everything becomes simpler.

The results here obtained give indications on how to structure the non-relativistic model. In fact, the classical model with an arbitrary number of moments known in literature proposes only 3 hierarchies with infinite equations. It is not shown how to interrupt these 3 blocks in order to obtain a finite system, except for the 2 simplest cases. This aspect is clarified here and, in particular, in section 2. The optimal choice of moments here presented in the subsystem with only one mode becomes the same as that already known in literature.

6. Declarations

6.1. Data Availability Statement

Data sharing is not applicable to this article.

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6.3. Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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